

Results of Paired-Domination of Some Special Graph Families on Transformation Graphs: G^{xy+} and G^{xy-}

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Abstract — In this study, transformation graphs obtained from the concept of the total graph and the result of its paired domination number for some special graph families are discussed. If a subset S of the vertex set of the graph G dominates and the induced subgraph $\langle S \rangle$ has a perfect matching that covers every vertex of the graph, then S is called a paired-dominating set of G . A paired dominating set with the smallest cardinality is denoted by γ_{pr} -set. Haynes and Slater introduced paired domination parameters. The present study commences with assessing outcomes stemming from eight permutations within the realm of path graphs. Subsequently, building upon this foundational structure, the results are extrapolated from the realm of cycle transformation graph structures based on findings from path transformation graphs.

Keywords Graph theory, graph vulnerability, paired domination number, transformation graphs

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1. Introduction

In graph theory, the investigation of domination numbers and their various forms has garnered significant attention, leading to a continual expansion of knowledge in this domain. Let G be a graph with the vertex set $V(G)$ (or simply V) and the edge set $E(G)$ (or simply E). Then, $S \subseteq V$ is called the dominating set of G if every vertex in $V - S$ is adjacent to any vertex in S , and the domination number of G is denoted by $\gamma(G)$. The domination number is the minimum cardinality of a dominating set.

Furthermore, the discourse extends to exploring the total domination number, a parameter that has drawn extensive discussion. A total dominating set S of G , denoted by TD -set, is a dominating set with no isolated vertex and total domination number, which is the minimum cardinality of a total dominating set abbreviated by $\gamma_t(G)$. Total domination was introduced by Cockayne et al. [1].

This paper delves into another alteration of the domination parameter, termed paired domination. Let $S \subseteq V$. Then, S is called a paired-dominating set of G and is denoted PD -set, if it dominates, and the induced subgraph $\langle S \rangle$ has a perfect matching that covers every vertex of the graph. A paired dominating set with the smallest cardinality is denoted by γ_{pr} -set. This concept was pioneered by Haynes and Slater [2]. To explain the meaning of these three domination parameters in terms of application, consider each $s \in S$ as a protector that can guard each vertex where s dominates then every protector guards itself. Regarding total domination, every protector must be guarded by another protector. As for paired domination, the guards' positions must be chosen as pairs of adjacent vertices so that each guard is assigned to each other and exists as substitutes for each other. There

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are many domination parameters in different categories, and the paired domination parameters discussed in this paper are considered under the category with restricted forms based on $\langle S \rangle$. We can present the reader with references to some significant studies, particularly in this work, that address the behavior of the paired domination number under various graph operations, as [3-11] in the literature. In addition, regarding the domination parameter, the three new books, which provide even more comprehensive coverage of current studies on domination parameters, can be offered to the reader as references [12-14].

In this paper, we consider simple graphs and refer to the book [15] for notation and terminology. The open neighborhood of any vertex in $V(G)$, denoted by $N(v) = \{u \in V(G) \mid (uv) \in E(G)\}$. The degree of a vertex v is the number of edges that are incident to it and is denoted by $\deg(v)$. Moreover, a vertex whose degree is one is called a pendant vertex. Furthermore, the maximum degree of G , denoted by $\Delta(G)$, is the degree of the vertex with the maximum number of edges incident to it. Similarly, the minimum degree of G , abbreviated by $\delta(G)$, is the degree of a vertex with the minimum number of edges incident to it. The distance between two vertices u and v is the shortest path between them and is denoted by the notation $d(u, v)$.

In this study, transformation graphs G^{xy-} and G^{xy+} are considered, and values of their paired-domination number for some special graph families are obtained. Transformation graph definitions are bottomed on the total graph [16]. Let $V(G) \cup E(G)$ be the vertex set of a total graph where two vertices are adjacent if and only if the vertices and edges are adjacent or incident in G . The transformation graphs introduced by Wu and Meng can be provided, denoted by G^{xyz} . Their paper considers its eight structure cases and generalizations of the total graph concept [17]. Before defining transformation graphs, we must explain some fundamental graph constructions such as complement and line graphs. The complement graph of G , denoted by the notation $\overline{G}$, is based on the same vertex set $V(G)$ but a different edge set completes the graph G to complement the graph. The line graph $L(G)$ of G is the graph that takes the edge set $E(G)$ as its vertex set. Two vertices are adjacent in the line graph $L(G)$ if and only if corresponding edges are adjacent in graph G [18]. Thus, the concept of the transformation graph represented by G^{xyz} can now be addressed, where x, y , and z can take values $+$ or $-$. The transformation graph G^{xyz} of G is a simple graph having as the vertex set $V(G) \cup E(G)$, and for $\alpha, \beta \in V(G) \cup E(G)$ and their relationships in G^{xyz} can be explained as follows:

- i. Let $\alpha, \beta \in V(G)$. Then, α and β are adjacent in G^{xyz} if $x = +$; otherwise $x = -$.
- ii. Let $\alpha, \beta \in E(G)$. Then, α and β are adjacent in G^{xyz} if $y = +$; otherwise $y = -$.
- iii. Let $\alpha \in V(G)$ and $\beta \in E(G)$. Then, α and β are incidents in G^{xyz} if $z = +$; otherwise $z = -$.

There are eight distinct constructions from three signature permutations $\{+, -\}$. Then, it may be obtained transformation graphs where G^{+++} is the total graph of G , and G^{---} is the complement of it. Moreover, G^{-+-} , G^{--+} , and G^{-++} are the complements of G^{+-+} , G^{+--} , and G^{+--} , respectively. Moreover, the transformation graph can be expressed as a relationship between G (or $\overline{G}$) and its line graph $L(G)$ (or $\overline{L(G)}$) depending on the sign of x, y , and z . If $x = +$, then the transformation graph includes G as a subgraph, otherwise includes $\overline{G}$ as a subgraph, and if $y = +$, then the transformation graph includes $L(G)$ as a subgraph, otherwise includes $\overline{L(G)}$ as a subgraph alike. Thus, depending on the sign of z , the relationship between G (or $\overline{G}$) and $L(G)$ (or $\overline{L(G)}$) has been structured based on the aforesaid situation (iii). Let $V(G) \cup V(L(G))$ be the vertex set of the transformation graph where $V(G) = \{1, 2, \dots, n\}$ and $V(L(G)) = \{(ij) : (ij) \in E(G)\}$. The vertex set will be expressed in this form unless otherwise stated. We direct readers' attention to select papers that offer comprehensive insights into the outcomes concerning specific domination parameters in transformation graphs. These referenced works provide detailed and in-depth information that can facilitate a deeper understanding of the subject matter of transformation graphs [19- 26].

2. Preliminaries

This section presents pivotal outcomes from scholarly literature concerning the paired domination number.

Proposition 2.1. [2] If u is adjacent to an end vertex of G , then u is in every paired dominating set. Furthermore, for certain specific families of graphs, $\gamma_{pr}(K_n) = \gamma_{pr}(K_{r,s}) = \gamma_{pr}(W_n) = 2$ and $\gamma_{pr}(P_n) = \gamma_{pr}(C_n) = 2 \left\lceil \frac{n}{4} \right\rceil$.

Here, the notation $\lceil \cdot \rceil$ represents the ceiling function.

Theorem 2.2. [2] For any graph G , $\gamma_{pr}(\overline{G}) \geq 4$ if and only if $\text{diam}(G) = 2$.

Theorem 2.3. [2] If a graph G has no isolated vertices, then $2 \leq \gamma_{pr}(G) \leq n$.

Proposition 2.4. [2] If a graph G has no isolated vertices, then $\gamma(G) \leq \gamma_t(G) \leq \gamma_{pr}(G)$, and $\gamma_{pr}(G)$ is even.

3. Results

This section presents the results of the eight different permutations of path and cycle transformation graphs for the paired domination parameter.

3.1. Paired Domination of Transformation Path Graphs

In this designated section, we intend to acquire and elucidate the outcomes of the paired domination of transformation graphs across all permutations. We shall commence our exposition by focusing on the specific P_n^{+++} form of permutations.

Theorem 3.1. Let $n > 6$ and P_n be a path graph. Then,

$$\gamma_{pr}(P_n^{+++}) = \begin{cases} \left\lceil \frac{4n-2}{7} \right\rceil, & n \equiv 0,3,4,6 \pmod{7} \\ \left\lceil \frac{4n-2}{7} \right\rceil + 1, & \text{otherwise} \end{cases}$$

PROOF.

The vertices of the graph P_n^{+++} can be partitioned into two parts, such as $V = V_1 \cup V_2$. To alleviate the burden of complex notation, the vertex sets can be labeled as $V_1 = \{1, 2, \dots, n\}$ and $V_2 = \{n+1, n+2, \dots, 2n\}$. Let D be a PD -set of P_n^{+++} . To begin, we establish an upper bound for $\gamma_{pr}(P_n^{+++})$ as follows: Let $S = S_1 \cup S_2$ be a set where S_1 includes the vertices of $V(P_n^{+++})$, and S_2 includes the vertices of $V(L(P_n^{+++}))$ as follows:

$$S_1 = \bigcup_{i=0}^{\left\lceil \frac{n-2}{7} \right\rceil - 1} \{(7i+2, 7i+3)\}$$

and

$$S_2 = \bigcup_{i=0}^{\left\lceil \frac{n-6}{7} \right\rceil - 1} \{(n+7i+5, n+7i+6)\}$$

Here, the cardinality of S is $|S| = 2 \left\lceil \frac{n-2}{7} \right\rceil + 2 \left\lceil \frac{n-6}{7} \right\rceil$.

i. If $n \equiv 0,3,4 \pmod{7}$, then $D = S$.

ii. If $n \equiv 1,2 \pmod{7}$, then $D = S \cup \{n, n-1\}$.

iii. If $n \equiv 5,6 \pmod{7}$, then $D = S \cup \{2n-2, 2n-1\}$.

Therefore, $|D| = \left\lceil \frac{4n-2}{7} \right\rceil$, for $n \equiv 0,3,4 \pmod{7}$, otherwise $|D| = \left\lceil \frac{4n-2}{7} \right\rceil + 1$. Thus,

$$\gamma_{pr}(P_n^{+++}) \leq \begin{cases} \left\lceil \frac{4n-2}{7} \right\rceil, & n \equiv 0,3,4 \pmod{7} \\ \left\lceil \frac{4n-2}{7} \right\rceil + 1, & \text{otherwise} \end{cases}$$

In order to prove the necessary condition, assume that the set $T = \{a_1, a_2, \dots, a_t\}$ is γ_{pr} -set of P_n^{+++} . Assume that the elements of T are ordered as $a_1 < a_2 < \dots < a_i < \dots < a_m < a_{m+1} < \dots < a_j < \dots < a_t$. Here, a_i and a_j are positive integers, $1 \leq a_i \leq n$, for all $i \in \{1, \dots, m\}$, and $n+1 \leq a_j \leq 2n-1$, for all $j \in \{m+1, \dots, t\}$. Define a function such as $f_x = a_{x+2} - a_x$ where $x \in \{1, 2, \dots, t-2\}$ such that $x \neq m$ and $x \neq m-1$. It is needed to prove that $f_x \leq 7$. Assume that a value of f_x is greater than or equal to 8. Without loss of generality, choose $f_1 = 8$. Therefore, for $n \geq 14$, we can create a set

$$S' = \{2, 3, n+5, n+6, n+7, n+8\} \cup \left(\bigcup_{i=0}^{\left\lceil \frac{n-10}{7} \right\rceil - 1} \{7i+10, 7i+11\} \right) \cup \left(\bigcup_{i=0}^{\left\lceil \frac{n-10}{7} \right\rceil - 1} \{n+7i+13, n+7i+14\} \right)$$

For $n \equiv 3 \pmod{7}$, $T = S' \cup \{n, n-1\}$. Thus,

$$|T| = |S'| + 2 = 8 + 2 \left(\frac{n-10}{7} \right) + 2 \left(\frac{n-10}{7} \right) = \frac{4n+16}{7}$$

However, it contradicts the obtained upper bound $\left\lceil \frac{4n-2}{7} \right\rceil$. Hence, f_x should be less than or equal to 7. Then, the upper bound for the total value of f_x can be gotten as follows:

$$\sum_{x_1=1}^{m-2} f_{x_1} + \sum_{x_2=m+1}^{t-2} f_{x_2} \leq 7(t-4)$$

In order to construct a relationship between the values of t and n , the total value of f_x can be evaluated as

$$\sum_{x_1=1}^{m-2} f_{x_1} + \sum_{x_2=m+1}^{t-2} f_{x_2} = a_m + a_{m-1} + a_t + a_{t-1} - (2n+16)$$

where $a_1 = 2$, $a_2 = 3$, $a_{m+1} = n+5$, and $a_{m+2} = n+6$.

If $n \equiv 0 \pmod{7}$, then

$$\sum_{x_1=1}^{m-2} f_{x_1} + \sum_{x_2=m+1}^{t-2} f_{x_2} = 4n - 28 \leq 7(t-4)$$

It is obtained that $t \geq \frac{4n}{7}$ where $a_m = n-4$, $a_{m-1} = n-5$, $a_t = 2n-1$, and $a_{t-1} = 2n-2$. Therefore,

$$|T| = t \geq \frac{4n}{7} = \left\lceil \frac{4n-2}{7} \right\rceil$$

Thus, it requires that $\gamma_{pr}(P_n^{+++}) \geq \left\lceil \frac{4n-2}{7} \right\rceil$, for $n \equiv 0 \pmod{7}$.

If $n \equiv 1 \pmod{7}$, then

$$\sum_{x_1=1}^{m-2} f_{x_1} + \sum_{x_2=m+1}^{t-2} f_{x_2} = 4n - 22 \leq 7(t-4)$$

It is obtained that $t \geq \frac{4n+6}{7}$ where $a_m = n$, $a_{m-1} = n-1$, $a_t = 2n-2$, and $a_{t-1} = 2n-3$. Therefore,

$$|T| = t \geq \frac{4n+6}{7} = \left\lceil \frac{4n-2}{7} \right\rceil + 1$$

Thus, $\gamma_{pr}(P_n^{+++}) \geq \left\lceil \frac{4n-2}{7} \right\rceil + 1$, for $n \equiv 1 \pmod{7}$.

If $n \equiv 2 \pmod{7}$, then

$$\sum_{x_1=1}^{m-2} f_{x_1} + \sum_{x_2=m+1}^{t-2} f_{x_2} = 4n - 24 \leq 7(t-4)$$

It is obtained that $t \geq \frac{4n+4}{7}$ where $a_m = n$, $a_{m-1} = n-1$, $a_t = 2n-3$, and $a_{t-1} = 2n-4$. Therefore,

$$|T| = t \geq \frac{4n+4}{7} = \left\lceil \frac{4n-2}{7} \right\rceil + 1$$

Thus, $\gamma_{pr}(P_n^{+++}) \geq \left\lceil \frac{4n-2}{7} \right\rceil + 1$, for $n \equiv 2 \pmod{7}$.

If $n \equiv 3 \pmod{7}$, then

$$\sum_{x_1=1}^{m-2} f_{x_1} + \sum_{x_2=m+1}^{t-2} f_{x_2} = 4n - 26 \leq 7(t-4)$$

It is obtained that $t \geq \frac{4n+2}{7}$ where $a_m = n$, $a_{m-1} = n-1$, $a_t = 2n-4$, and $a_{t-1} = 2n-5$. Therefore,

$$|T| = t \geq \frac{4n+2}{7} = \left\lceil \frac{4n-2}{7} \right\rceil$$

Thus, $\gamma_{pr}(P_n^{+++}) \geq \left\lceil \frac{4n-2}{7} \right\rceil$, for $n \equiv 3 \pmod{7}$.

If $n \equiv 4 \pmod{7}$, then

$$\sum_{x_1=1}^{m-2} f_{x_1} + \sum_{x_2=m+1}^{t-2} f_{x_2} = 4n - 30 \leq 7(t-4)$$

It is obtained that $t \geq \frac{4n-2}{7}$ where $a_m = n-1$, $a_{m-1} = n-2$, $a_t = 2n-5$, and $a_{t-1} = 2n-6$. Therefore,

$$|T| = t \geq \frac{4n-2}{7} = \left\lceil \frac{4n-2}{7} \right\rceil$$

Thus, $\gamma_{pr}(P_n^{+++}) \geq \left\lceil \frac{4n-2}{7} \right\rceil$, for $n \equiv 4 \pmod{7}$.

If $n \equiv 5 \pmod{7}$, then

$$\sum_{x_1=1}^{m-2} f_{x_1} + \sum_{x_2=m+1}^{t-2} f_{x_2} = 4n - 24 \leq 7(t-4)$$

It is obtained that $t \geq \frac{4n+4}{7}$ where $a_m = n-2$, $a_{m-1} = n-3$, $a_t = 2n-1$, and $a_{t-1} = 2n-2$. Therefore,

$$|T| = t \geq \frac{4n+4}{7} = \left\lceil \frac{4n-2}{7} \right\rceil + 1$$

Thus, $\gamma_{pr}(P_n^{+++}) \geq \left\lceil \frac{4n-2}{7} \right\rceil + 1$, for $n \equiv 5 \pmod{7}$.

If $n \equiv 6 \pmod{7}$, then

$$\sum_{x_1=1}^{m-2} f_{x_1} + \sum_{x_2=m+1}^{t-2} f_{x_2} = 4n - 26 \leq 7(t - 4)$$

It is obtained that $t \geq \frac{4n+2}{7}$ where $a_m = n - 3$, $a_{m-1} = n - 4$, $a_t = 2n - 1$, and $a_{t-1} = 2n - 2$. Therefore,

$$|T| = t \geq \frac{4n+2}{7} = \left\lceil \frac{4n-2}{7} \right\rceil$$

Thus, $\gamma_{pr}(P_n^{+++}) \geq \left\lceil \frac{4n-2}{7} \right\rceil + 1$ for $n \equiv 6 \pmod{7}$.

Hence, from the necessary and sufficient conditions, we obtain the result as

$$\gamma_{pr}(P_n^{+++}) = \begin{cases} \left\lceil \frac{4n-2}{7} \right\rceil, & n \equiv 0, 3, 4, 6 \pmod{7} \\ \left\lceil \frac{4n-2}{7} \right\rceil + 1, & \text{otherwise} \end{cases}$$

Theorem 3.2. Let $n > 5$ and P_n be a path graph. Then, $\gamma_{pr}(P_n^{++-}) = 4$.

PROOF.

Consider the set of vertices $\{1, 2, \dots, n\}$ as the vertex set of P_n . Let S be paired dominating set of P_n^{++-} and $S = S_1 \cup S_2$ where $S_1 = \{u \mid u \in V(P_n)\}$ and $S_2 = \{e \mid e \in V(L(P_n))\}$. If the vertex 1 is part of S_1 then all the vertices in $L(P_n)$ except for $(12) \in V(L(P_n))$ are dominated. To achieve perfect matching and ensure the domination of the vertex (12) and the remaining vertices $\{3, 4, \dots, n\} \in V(P_n)$, it is needed to add a vertex $e = (uv)$ to S_2 , which is adjacent to 1, and v and $v + 1$ to S_1 . Therefore, $\{1, (uv), v, v + 1\}$ is a paired dominating set for P_n^{++-} and $|S| \leq 4$. However, it is impossible to dominate all the vertices with two pairs of vertices. Then, $|S| > 2$. Moreover, since the paired domination number must be an even number according to Proposition 2.4. Then,

$$\gamma_{pr}(P_n^{++-}) = 4$$

□

Theorem 3.3. Let $n \geq 6$ and P_n be a path graph. Then, $\gamma_{pr}(P_n^{+-+}) = 2 + \gamma_{pr}(P_{n-4})$.

PROOF.

Consider the set of vertices $\{1, 2, \dots, n\}$ as the vertex set of P_n . Let S be the minimum paired dominating set of P_n^{+-+} . It is known that $|S| \geq 2$ from Theorem 2.3. If the vertices (12) and $((n-1)n)$ with the maximum degree in P_n^{+-+} are added to S , then all the vertices in $V(\overline{L(P_n)})$ are dominated. However, there are some non-dominated vertices such as $\{3, 4, \dots, (n-2)\} \in V(P_n^{+-+})$. These non-dominated $(n-4)$ vertices construct a P_{n-4} structure. As a result, the size of S can be expressed as $2 + \gamma_{pr}(P_{n-4})$. □

Theorem 3.4. Let P_n be a path graph. Then, $\gamma_{pr}(P_n^{+--}) = 2$.

PROOF.

Let $\{1, 2, \dots, n\}$ be the vertex set of P_n . Let S be a paired dominating set of P_n^{+--} . It is clear that all the vertices in P_n^{+--} dominated pairedly by $\{(12), ((n-1)n)\}$. This insight leads to the conclusion that $|S| \leq 2$. By Theorem 2.3, it is established that S must have a size of at least 2. Consequently, combining these observations, we can affirm that $|S| = 2$. □

Theorem 3.5. Let P_n be a path graph. Then, $\gamma_{pr}(P_n^{---}) = 2$.

PROOF.

Let $\{1, 2, \dots, n\}$ be the vertex set of P_n . Let S be a paired dominating set of P_n^{---} . As in the proof of P_n^{+--} , all the vertices in P_n^{---} dominated pairedly by $\{(12), ((n-1)n)\}$. Therefore, $|S| \leq 2$. It is known that $|S| \geq 2$ from Theorem 2.3. Consequently, $|S| = 2$. $\square$

Theorem 3.6. Let P_n be a path graph. Then, $\gamma_{pr}(P_n^{--+}) = 4$.

PROOF.

Let $\{1, 2, \dots, n\}$ be the vertex set of P_n . Consider that S be paired dominating set of P_n^{--+} and $S = S_1 \cup S_2$ where $S_1 = \{u \mid u \in V(P_n)\}$ and $S_2 = \{e \mid e \in V(L(P_n))\}$. If $1 \in S_1$, then all the vertices in P_n^{--+} except for $2 \in V(P_n)$ and the vertex $(12) \in V(L(P_n))$ are dominated. In order to get perfect matching and dominate $2 \in V(P_n)$, it is needed to add a vertex (12) to S_2 , which is adjacent to 1. However, the vertex (23) is the only non-dominated vertex in the graph. Therefore, a pair should be added to the set as $\{1, (12), v, v+1\}$, a paired dominating set for P_n^{--+} . Hence, $|S| \leq 4$. Furthermore, it is not feasible to achieve domination over all the vertices using only two pairs of vertices, leading to the conclusion that $|S| > 2$. Considering Proposition 2.4 stating the paired domination number must be an even value, $\gamma_{pr}(P_n^{--+}) = 4$. $\square$

Theorem 3.7. Let P_n be a path graph. Then, $\gamma_{pr}(P_n^{+-}) = 2$.

PROOF.

Consider the set of vertices $\{1, 2, \dots, n\}$ as the vertex set of P_n . Let S represent the paired dominating set of P_n^{+-} . All the vertices in P_n^{+-} are dominated pairedly by $\{1, n\}$. As a direct consequence, we deduce that $|S| \leq 2$. It is established in Theorem 2.3 that $|S|$ must have a minimum size of 2. Hence, combining these results, $|S| = 2$. $\square$

Theorem 3.8. Let P_n be a path graph. Then, $\gamma_{pr}(P_n^{++}) = 2 + \gamma_{pr}(P_{n-5})$.

PROOF.

Let $\{1, 2, \dots, n\}$ be the vertex set of P_n , and S be the minimum paired dominating set of P_n^{++} . It is known that $|S| \geq 2$ from Theorem 2.3. Upon introducing the vertices (2) and $((n-1))$, which possess the highest degree in G , are added to the set S , then all the vertices in $V(\overline{P_n})$ are dominated. Nevertheless, there are some non-dominated vertices, such as $\{(34), (45), \dots, ((n-3)(n-2))\}$ in $V(L(G))$. These non-dominated $(n-5)$ vertices construct a P_{n-5} structure. Therefore, it is obtained that $S = 2 + \gamma_{pr}(P_{n-5})$. $\square$

3.2. Paired Domination of Transformation Cycle Graphs

Given the particular structure of the cycle graph, it is noteworthy that the line graph of C_n exhibits isomorphism to itself. This observation leads to deduce the subsequent results:

Theorem 3.9. Let C_n be a cycle graph. Then, $\gamma_{pr}(C_n^{+-}) = 4$.

PROOF.

Let $\{1, 2, 3, \dots, n\}$ be the vertex set of C_n , and S be the paired dominating set of C_n^{+-} . If S contains the vertex 1 and the vertex (12) , then it ensures the domination of all the vertices within the graph. Nevertheless, these two vertices are not adjacent according to the form of the graph. In order to get perfect matching, it is needed to add two more vertex. It holds that the size of set S exceeds 2. Consequently, we can deduce that the parameter $\gamma_{pr}(C_n^{+-})$ is equal to 4. $\square$

Theorem 3.10. Let C_n be a cycle graph and $n \geq 9$. Then, $\gamma_{pr}(C_n^{++}) = \gamma_{pr}(C_n^{+-}) = 4 + 2 \left\lceil \frac{n-8}{4} \right\rceil$.

PROOF.

Let $\{1, 2, 3, \dots, n\}$ be the vertex set of C_n and S be the paired dominating set of C_n^{-++} . If S includes the vertices (12) , $(1n)$, 4 , and $(n-2)$, then all the vertices of $\overline{C_n}$ and the vertices labeled by (12) , (23) , (34) , (45) , $((n-3)(n-2))$, $((n-2)(n-1))$, $((n-1)(n))$, and $(n1)$ in $L(C_n)$ are dominated pairedly. However, the non-dominated P_{n-8} the structure remains. Consequently, from [2]

$$\gamma_{pr}(C_n^{-++}) = \gamma_{pr}(C_n^{+-+}) = 4 + \gamma_{pr}(P_{n-8}) = 4 + 2 \left\lfloor \frac{n-8}{4} \right\rfloor$$

□

Theorem 3.11. Let C_n be a cycle graph and $n \geq 6$. Then, $\gamma_{pr}(C_n^{-+-}) = \gamma_{pr}(C_n^{+--}) = 2$.

PROOF.

Consider the vertex set $\{1, 2, 3, \dots, n\}$ as the vertex set of C_n , and let S denote a paired dominating set of C_n^{+--} . If S includes (12) and (45) from $V(L(C_n))$, then all the vertices in C_n^{+--} are dominated pairedly. Consequently, it follows that $\gamma_{pr}(C_n^{-+-}) = \gamma_{pr}(C_n^{+--}) = 2$, establishing the paired domination number for C_n^{-+-} and C_n^{+--} . □

Theorem 3.12. Let C_n be a cycle graph and $n \geq 6$. Then, $\gamma_{pr}(C_n^{--+}) = 4$.

PROOF.

Given the graph C_n^{--+} , the complete graph of C_n^{++-} , it can be deduced from Theorem 2.2 that its diameter is 2. Consequently, this implies that $\gamma_{pr}(C_n^{--+}) \geq 4$. Considering the vertex set $\{1, 2, 3, \dots, n\}$ as the vertex set of C_n , and let S be paired dominating set of C_n^{--+} . If S is included (1) and (3) from $V(\overline{C_n})$ and (12) and (34) from $V(L(C_n))$, then all the vertices in C_n^{--+} are dominated. This observation leads to the conclusion that $|S| = \gamma_{pr}(C_n^{--+}) \leq 4$. Hence, $\gamma_{pr}(C_n^{--+}) = 4$. □

Theorem 3.13. Let C_n be a cycle graph. Then, $\gamma_{pr}(C_n^{---}) = 2$.

PROOF.

Consider the vertex set $\{1, 2, 3, \dots, n\}$ of the C_n , and let S be a paired dominating set of C_n^{---} . If S contains the vertices 1 and 4 , then it ensures that S dominates $\overline{C_n}$ and $\overline{L(C_n)}$. Additionally, $\langle S \rangle$ forms a graph with perfect matching. □

Theorem 3.14. Let C_n be a cycle graph. Then,

$$\gamma_{pr}(C_n^{+++}) = \begin{cases} \gamma_{pr}(P_n^{+++}) + 2, & n \equiv 4 \pmod{7} \\ \gamma_{pr}(P_n^{+++}), & \text{otherwise} \end{cases}$$

PROOF.

Let S be the γ_{pr} -set of P_n^{+++} . Considered similar to the proof of the P_n^{+++} , S is also the γ_{pr} -set of C_n^{+++} when $n \not\equiv 4 \pmod{7}$. If the vertex set of $L(C_n)$ labeled as in the proof of P_n^{+++} , the vertex $(n1)$ corresponds to the vertex $2n$. However, the vertex $(2n) \in V(L(C_n^{+++}))$ cannot be dominated by S when $n \equiv 4 \pmod{7}$. In such cases, the addition of two extra vertices becomes necessary to ensure paired domination. Hence,

$$\gamma_{pr}(C_n^{+++}) = \gamma_{pr}(P_n^{+++}) + 2$$

when $n \equiv 4 \pmod{7}$. Henceforth, it is deduced that

$$\gamma_{pr}(C_n^{+++}) = \begin{cases} \gamma_{pr}(P_n^{+++}) + 2, & n \equiv 4 \pmod{7} \\ \gamma_{pr}(P_n^{+++}), & \text{otherwise} \end{cases}$$

□

4. Conclusion

We have implemented the discussed paired domination parameter from this research on path and cycle structures, two significant classes within the realm of graphs. We have derived outcomes for the transformation graphs encompassing all eight permutations in both these structural contexts. The outcome $\gamma_{pr}(P_n^{+++})$ is then applied to derive the $\gamma_{pr}(C_n^{+++})$ result for the cycle structure.

In upcoming research endeavors related to the topic, paired domination values of transformation graphs can be investigated for a general G graph. Furthermore, an attempt can be made to establish associations with other graph parameters to derive lower or upper-bound values. Since the concept of domination holds a significant position in graph theory, researchers interested in the subject might also explore the influence of transformation graphs on other domination parameters.

Author Contributions

The author read and approved the final version of the paper.

Conflict of Interest

The author declares no conflict of interest.

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