

Photon in a cylindrical resonant cavity

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Received: 22.08.2017

Accepted/Published Online: 29.11.2017

Final Version: 26.04.2018

Abstract: In this study, the massless and massive spin-1 particle equations, derived from the excited states of the zitterbewegung model, are considered for the photon in the cylindrical resonant cavity background. The resonant frequencies of the particles are also obtained. We show that these frequencies become equivalent in the $M^2 \rightarrow 0$ limit.

Key words: Zitterbewegung model, massless and massive spin-1 particle, photon equation, Duffin–Kemmer–Petiau equation, resonant cavity

1. Introduction

Humankind has been illuminated about the mysteries of light as a result of the essential contributions of theoretical and experimental research, which has attempted to comprehend its nature over many decades. The crucial enlightenment began with the classical approach, which describes light in terms of an electromagnetic field and finite speed using four equations [1], and it continued with the idea that the Maxwell equations were relativistic wave equations [2]. High-energy particles can be characterized by a relativistic particle equation by virtue of the quantum mechanics that emerges when the wavelengths of particles are comparable to their sizes. Thus, different relativistic quantum mechanical equations have been proposed to describe the elementary particles since it was understood that they have a feature called spin. Some of the proposed relativistic particle equations are the Weyl equation for massless spin-1/2 particles [3], the Dirac equation for massive spin-1/2 particles (which can be reduced to the Weyl equation in the $M^2 \rightarrow 0$ limit [4]), the Rarita–Schwinger equation for spin-3/2 particles [5], the Klein–Gordon equation for spin-0 particles [6, 7], and the Duffin–Kemmer–Petiau (DKP) equation for both spin-0 and spin-1 particles [8–10]. Even though the solutions of all these equations have been investigated in various spacetime backgrounds, this paper focuses only on the quantum electrodynamics (QED) of the photon as a spin-1 particle. The equivalence of the spin-1 part of the DKP equation to the classical Maxwell equations [11] can provide important insights due to the different approach of QED compared to classical physics. In this context, the massive spin-1 particle equation is derived from the excited states of the zitterbewegung model, which is essentially equivalent to the spin-1 part of the DKP equation in flat spacetime [12]. Subsequently, its massless case has been represented as a toy model of the zitterbewegung [13]. Afterwards, the massless and the massive spin-1 particle equations were generalized to curved spacetime [14,15]. The symmetry and the integrability properties of the zitterbewegung model were also investigated in $1+1$, $2+1$, and $3+1$ dimensional spacetimes [16]. Yet other studies of the spin-1 particle have been performed to calculate

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the Hawking radiation by means of the quantum spin-1 tunneling method [17,18] and the Noether charge in $2+1$ dimensional spacetime [19].

The physical behavior of an electromagnetic field in various resonant cavities has been investigated both theoretically and experimentally [20]. On the other hand, since Planck's quantum hypothesis, we know that electromagnetic radiation is composed of the $h\nu$ energized quanta called photons; a photon is a quantum of the electromagnetic field [21]. However, the quantum dynamics of the photon in a cylindrical resonant cavity has not been considered so far. Therefore, we deal with the quantum electrodynamical behavior of the photon in a cylindrical resonant cavity by using both the massless and massive spin-1 particle equations, which are derived from the excited states of the zitterbewegung.

An outline of the study is given as follows: the massless spin-1 particle equation is represented and solved in a cylindrical resonant cavity in Section 2; in Section 3 a similar procedure is followed for the massive spin-1 equation. Finally, we discuss the resonant frequencies by comparing the massless case with the massive case in the $M^2 \rightarrow 0$ limit.

2. The massless spin-1 particle in a cylindrical resonant cavity

The massless spin-1 particle equation was derived from the toy model of the zitterbewegung [13]. Its covariant form in a curved spacetime is

$$\{i\hbar\Sigma^\mu(x)[\partial_\mu - \Gamma_\mu(x) \otimes I - I \otimes \Gamma_\mu(x)]\}_{\gamma\delta}\psi_{\gamma\delta}(x) = 0 \quad (1)$$

where $\Sigma^\mu(x)$ is defined as $\sigma^\mu(x) \otimes I + I \otimes \sigma^\mu(x)$ and $\psi_{\gamma\delta}(x)$ is the 4×1 symmetric spinor. The spin connection for spin-1/2, $\Gamma_\mu(x)$, is defined by [14] as follows:

$$\Gamma_\mu(x) = -\frac{1}{8}[\sigma^\nu(x), \sigma_{\nu;\mu}(x)] \quad (2)$$

where $\sigma^\nu(x)$ are the spacetime dependent Pauli matrices and are obtained from the following relation:

$$\sigma^\mu(x) = e_a^\mu(x) \sigma^a \quad (3)$$

where σ^a are the constant Pauli matrices and $e_a^\mu(x)$ are tetrads satisfying the relation, $e_a^\mu(x) e_b^\nu(x) \eta^{ab} = g^{\mu\nu}$.

The metric tensor of the cylindrical resonant cavity background is $g_{\mu\nu} = \text{diag}[-1, -r^2, -1, 1]$. The tetrads can then be written as $e_b^\mu(x) = \text{diag}[1, \frac{1}{r}, 1, 1]$. Therefore, the nonzero spin connection in this background is

$$\Gamma_2(x) = -\frac{i}{2}\sigma_3 \quad (4)$$

Using the spin connection, the massless spin-1 particle equation in the cylindrical resonant cavity can be written as

$$\left[\vec{\Sigma} \cdot \vec{\nabla} + 2(I \otimes I) \partial_t - \frac{i}{2r} \Sigma_2 \Sigma_3 \right] \psi = 0 \quad (5)$$

Thanks to the separation of variables method, the 4×1 spinor ψ for the cylindrical resonant cavity is defined as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = e^{i(-\omega t + kz + m\phi)} \begin{pmatrix} R_+ \\ R_0 \\ R_0 \\ R_- \end{pmatrix} \quad (6)$$

and we find the three first-order differential equations as follows:

$$\left(\frac{d}{dr} + \frac{m}{r}\right) R_0 - i\left(\frac{\omega}{c} - k\right) R_+ = 0, \quad (7)$$

$$\left(\frac{d}{dr} + \frac{1}{r}\right) (R_+ + R_-) - \frac{m}{r} (R_+ - R_-) - 2i\frac{\omega}{c} R_0 = 0, \quad (8)$$

$$\left(\frac{d}{dr} - \frac{m}{r}\right) R_0 - i\left(\frac{\omega}{c} + k\right) R_- = 0. \quad (9)$$

Then, by adding and subtracting these equations, we find the Bessel differential equation [22]:

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \left(u^2 - \frac{m^2}{r^2}\right)\right] R_0(r) = 0, \quad (10)$$

with solution

$$R_0(r) = N_1 J_m(ur) + N_2 Y_m(ur) \quad (11)$$

where N_1 and N_2 are integration constants and $u^2 = \frac{\omega^2}{c^2} - k^2$.

Using the Bessel recurrence relations [22] after replacing $R_0(r)$ in both Eq. (6) and Eq. (9), the complete solution is found as follows:

$$\psi(r, \phi, z, t) = e^{i(-\omega t + kz + m\phi)} \begin{pmatrix} -\frac{i}{u} \left(\frac{\omega}{c} + k\right) [N_1 J_{m-1}(ur) + N_2 Y_{m-1}(ur)] \\ N_1 J_m(ur) + N_2 Y_m(ur) \\ N_1 J_m(ur) + N_2 Y_m(ur) \\ \frac{i}{u} \left(\frac{\omega}{c} - k\right) [N_1 J_{m+1}(ur) + N_2 Y_{m+1}(ur)] \end{pmatrix} \quad (12)$$

where $J_m(ur)$ is the Bessel function and $Y_m(ur)$ is the Neumann function.

The boundary conditions defining the cylindrical resonant cavity are

$$\psi(r, \phi, z) = \begin{cases} 0, & r = 0 \quad \text{and} \quad r = a \\ 0, & z = 0 \quad \text{and} \quad z = L \end{cases}. \quad (13)$$

Using the boundary conditions, N_2 is chosen as zero due to the divergence of the Neumann function at $r = 0$ and

$$ua = \chi_{mn} \text{ at } r = a \quad (14)$$

where χ_{mn} are the zeros of the Bessel function. Moreover, e^{ikz} yields $\sin(kz)$ at $z = 0$, which defines the free particle solution in the z direction and $\sin(kL) = 0$ at $z = L$ condition gives the wave numbers $k = \frac{p\pi}{L}$ as $p \geq 0$ integers. Consequently, the solution of the cylindrical resonant cavity is rewritten as

$$\psi(r, \phi, z, t) = N_1 \sin\left(\frac{p\pi}{L} z\right) e^{i(-\omega t + m\phi)} \begin{pmatrix} -ik_- J_{m-1}(ur) \\ J_m(ur) \\ J_m(ur) \\ +ik_+ J_{m+1}(ur) \end{pmatrix} \quad (15)$$

where $k_{\pm} = \sqrt{1 + \left(\frac{k}{u}\right)^2} \pm \frac{k}{u}$.

3. The massive spin-1 particle in a cylindrical resonant cavity

The massive spin-1 particle equation, derived from the excited states of the zitterbewegung model in a curved spacetime, is given in its covariant form as

$$\{i\hbar\beta^\mu(x)[\partial_\mu - \Gamma_\mu(x) \otimes I - I \otimes \Gamma_\mu(x)] - m_0c\}_{\alpha\beta,\gamma\delta}\psi_{\gamma\delta}(x) = 0 \quad (16)$$

where the Kemmer matrices, $\beta^\mu(x)$, are defined as $[\gamma^\mu(x) \otimes I + I \otimes \gamma^\mu(x)]/2$, m_0 is the mass of the spin-1 particle, c is the speed of light in a vacuum, and $\psi_{\gamma\delta}(x)$ is the 16×1 symmetric spinor [15]. The spin connection for spin-1/2, $\Gamma_\mu(x)$, is defined by [15] as

$$\Gamma_\mu(x) = -\frac{1}{8}[\gamma^\nu(x), \gamma_{\nu;\mu}(x)] \quad (17)$$

where $\gamma^\nu(x)$ is the Dirac matrices in the general coordinate frame. These are transformed into a curved spacetime by means of the tetrads, $e_b^\mu(x)$, obtained from the following relation:

$$\gamma^\mu(x) = e_b^\mu(x) \gamma^b \quad (18)$$

and they satisfy anticommutation relation:

$$\gamma^\mu(x) \gamma^\nu(x) + \gamma^\nu(x) \gamma^\mu(x) = 2g^{\mu\nu} \quad (19)$$

The metric tensor of the cylindrical resonant cavity background and the tetrads are given in Section 2. Therefore, the nonzero spin connection in this background is

$$\Gamma_2(x) = \frac{i}{2} \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \quad (20)$$

Then the massive spin-1 particle equation in the cylindrical resonant cavity can be written as follows:

$$\left\{ \vec{\beta} \cdot \vec{\nabla} + \beta^4(x) \partial_t - \beta^2(x) [\Gamma_2(x) \otimes I + I \otimes \Gamma_2(x)] + iM \right\}_{\alpha\beta,\gamma\delta} \psi_{\gamma\delta}(x) = 0 \quad (21)$$

where M is defined as $m_0c/\hbar$. Using the separation of variables method, the 16×1 symmetric spinor, ψ , for the cylindrical resonant cavity is defined as

$$\psi(r, \phi, z, t) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = e^{i(-\omega t + kz + m\phi)} \begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} \quad (22)$$

$$R_1(r) = \begin{pmatrix} R_{1+} \\ R_{10} \\ R_{10} \\ R_{1-} \end{pmatrix}, \quad R_2(r) = \begin{pmatrix} R_{2+} \\ R_{20} \\ R_{2\bar{0}} \\ R_{2-} \end{pmatrix}, \quad R_3(r) = \begin{pmatrix} R_{2+} \\ R_{2\bar{0}} \\ R_{20} \\ R_{2-} \end{pmatrix}, \quad R_4(r) = \begin{pmatrix} R_{4+} \\ R_{40} \\ R_{40} \\ R_{4-} \end{pmatrix} \quad (23)$$

due to the cylindrical symmetry. Thus, the Hamiltonian in Eq. (21) can be written as

$$\begin{pmatrix} (2\partial_t + 2iM)I \otimes I & A(x) & B(x) & 0 \\ -A(x) & 2iM(I \otimes I) & 0 & B(x) \\ -B(x) & 0 & 2iM(I \otimes I) & A(x) \\ 0 & -B(x) & -A(x) & (-2\partial_t + 2iM)I \otimes I \end{pmatrix} \quad (24)$$

where $A(x) = I \otimes \vec{\sigma} \cdot \vec{\nabla} + \delta_1 I \otimes \sigma_2 \Sigma_3$ and $B(x) = \vec{\sigma} \cdot \vec{\nabla} \otimes I + \delta_1 \sigma_2 \otimes I \Sigma_3$. The four first-order differential equations are then written as

$$-2i\omega(R_1 - R_4) + 2iM(R_1 + R_4) - [\vec{\Sigma} \cdot \vec{\nabla} + \delta_1 \bar{\Sigma}_2 \Sigma_3](R_2 - R_3) = 0 \quad (25)$$

$$-2i\omega(R_1 + R_4) + 2iM(R_1 - R_4) + [\vec{\Sigma} \cdot \vec{\nabla} + \delta_1 \Sigma_2 \Sigma_3](R_2 + R_3) = 0 \quad (26)$$

$$2iM(R_2 + R_3) - [\vec{\Sigma} \cdot \vec{\nabla} + \delta_1 \Sigma_2 \Sigma_3](R_1 - R_4) = 0 \quad (27)$$

$$2iM(R_2 - R_3) + [\vec{\Sigma} \cdot \vec{\nabla} + \delta_1 \bar{\Sigma}_2 \Sigma_3](R_1 + R_4) = 0 \quad (28)$$

where $\vec{\Sigma} = \vec{\sigma} \otimes I - I \otimes \vec{\sigma}$ and $\delta_1 = -i/2r$. Adding and subtracting Eqs. (25) and (26) and Eqs. (27) and (28), we find transverse ($\pm$ helicity) states:

$$(R_1 + R_4)_\pm = \frac{i}{u^2} \left\{ \left(\frac{d}{dr} \pm \frac{m}{r} \right) [-\omega(R_{20} + R_{2\bar{0}}) \mp M(R_{20} - R_{2\bar{0}})] \pm 2ikMR_{2\pm} \right\} \quad (29)$$

$$(R_1 - R_4)_\pm = \frac{i}{u^2} \left\{ \left(\frac{d}{dr} \pm \frac{m}{r} \right) [\mp\omega(R_{20} - R_{2\bar{0}}) - M(R_{20} + R_{2\bar{0}})] \mp 2ikMR_{2\pm} \right\} \quad (30)$$

and longitudinal (zero helicity) states:

$$(R_1 + R_4)_0 = \frac{i}{u^2} \left\{ ikM(R_{20} - R_{2\bar{0}}) - \omega \left[\left(\frac{d}{dr} + \frac{1}{r} \right) (R_{2+} + R_{2-}) - \frac{m}{r} (R_{2+} - R_{2-}) \right] \right\} \quad (31)$$

$$(R_1 - R_4)_0 = \frac{i}{u^2} \left\{ -M \left[\left(\frac{d}{dr} + \frac{1}{r} \right) (R_{2+} + R_{2-}) - \frac{m}{r} (R_{2+} - R_{2-}) \right] + ik\omega(R_{20} - R_{2\bar{0}}) \right\} \quad (32)$$

where $u^2 = \frac{\omega^2}{c^2} - M^2$. After straightforward calculations, we find the following second-order differential equations:

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \left(\tilde{u}^2 - \frac{m^2}{r^2} \right) \right] (R_{20} \pm R_{2\bar{0}}) = 0 \quad (33)$$

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \left(\tilde{u}^2 - \frac{(m-1)^2}{r^2} \right) \right] R_{2+} = 0, \quad (34)$$

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \left(\tilde{u}^2 - \frac{(m+1)^2}{r^2} \right) \right] R_{2-} = 0 \quad (35)$$

where $\tilde{u}^2$ is defined as

$$\tilde{u}^2 = u^2 - k^2 = \chi_{mn}^2/a^2 \quad (36)$$

The solutions of Eqs. (33), (34), and (35) are found, respectively:

$$(R_{20} \pm R_{2\bar{0}}) = N_{\pm} J_m(\tilde{u}r) \quad (37)$$

$$R_{2+} = N_{2+} J_{m-1}(\tilde{u}r) \quad (38)$$

$$R_{2-} = N_{2-} J_{m+1}(\tilde{u}r) \quad (39)$$

where $N_{\pm}$, $N_{2\pm}$ are integration constants. For simplicity, the integration constants have values $N_{\pm} = N_{2\pm} = N$. Finally, the complete solutions are found as follows:

$$\psi_1 = N \sin\left(\frac{p\pi}{L}z\right) e^{i(-\omega t + m\phi)} \tilde{k}_+ \begin{pmatrix} \left(1 - i\frac{\tilde{u}}{k}\right) J_{m-1}(\tilde{u}r) \\ \frac{1}{2} J_m(\tilde{u}r) \\ \frac{1}{2} J_m(\tilde{u}r) \\ -J_{m+1}(\tilde{u}r) \end{pmatrix} \quad (40)$$

$$\psi_2 = N \sin\left(\frac{p\pi}{L}z\right) e^{i(-\omega t + m\phi)} \begin{pmatrix} J_{m-1}(\tilde{u}r) \\ J_m(\tilde{u}r) \\ 0 \\ J_{m+1}(\tilde{u}r) \end{pmatrix} \quad (41)$$

$$\psi_3 = N \sin\left(\frac{p\pi}{L}z\right) e^{i(-\omega t + m\phi)} \begin{pmatrix} J_{m-1}(\tilde{u}r) \\ 0 \\ J_m(\tilde{u}r) \\ J_{m+1}(\tilde{u}r) \end{pmatrix} \quad (42)$$

$$\psi_4 = N \sin\left(\frac{p\pi}{L}z\right) e^{i(-\omega t + m\phi)} \tilde{k}_- \begin{pmatrix} J_{m-1}(\tilde{u}r) \\ -\frac{1}{2} J_m(\tilde{u}r) \\ -\frac{1}{2} J_m(\tilde{u}r) \\ -\left(1 - i\frac{\tilde{u}}{k}\right) J_{m+1}(\tilde{u}r) \end{pmatrix} \quad (43)$$

where $\tilde{k}_{\pm} = \left[\sqrt{1 + \frac{\tilde{u}^2 + M^2}{k^2}} \pm \frac{M}{k} \right]^{-1}$.

4. Concluding remarks

In this study, the massless and massive spin-1 particle equations derived from the zitterbewegung model are solved in the cylindrical resonant cavity background. From these solutions we obtain expressions for the resonant

frequencies of the photon in the cylindrical resonant cavity background. From Eq. (14) for the massless case and from Eq. (36) for the massive case, we obtain resonance frequency relations, respectively:

$$\omega_{mnp} = \frac{c}{a} \sqrt{\chi_{mn}^2 + p^2 \pi^2 \left(\frac{a}{L}\right)^2} \quad (44)$$

and

$$\omega_{mnp} = \frac{c}{a} \sqrt{\chi_{mn}^2 + p^2 \pi^2 \left(\frac{a}{L}\right)^2 + M^2 a^2} \quad (45)$$

The resonant frequency relation for the massless case in Eq. (44) is equivalent to the classical result [20], while the resonant frequency relation for the massive case in Eq. (45) confirms this result in the $M^2 \rightarrow 0$ limit, but with m, n , and p being quantum numbers in this context. The case where the resonant frequency is affected by only the m and n quantum numbers occurs when the length of the cavity is much larger than the radius. In contrast, the case where the resonant frequency is affected by only the p quantum number occurs when the cavity length is much smaller than the radius.

Acknowledgment

This work was supported by the scientific research projects unit of Akdeniz University.

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