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Higher order generalized geometric polynomials

Levent KARGIN[1,*], **Bayram ÇEKİM**[2]
[1]Vocational School of Technical Sciences, Akdeniz University, Antalya, Turkey
[2]Department of Mathematics, Faculty of Sciences, Gazi University, Ankara, Turkey

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Abstract: According to the generalized Mellin derivative, we introduce a new family of polynomials called higher order generalized geometric polynomials and obtain some arithmetical properties of them. Then we investigate the relationship of these polynomials with degenerate Bernoulli, degenerate Euler, and Bernoulli polynomials. Finally, we evaluate several series and integrals in closed forms.

Key words: Generalized geometric polynomials, Bernoulli polynomials, Euler polynomials, Riemann zeta function

1. Introduction

The operator $\left(x\frac{d}{dx}\right)^n$, called the Mellin derivative [6], has a long mathematical history. As far back as 1740, Euler used the operator as a tool in his work [20]. After that, the Mellin derivative and its generalizations were used to obtain a new class of polynomials [5, 18, 19, 26, 34], to evaluate some power series in closed forms [5, 8, 18, 19, 26, 28, 34], and to calculate some integrals [6, 8]. One generalization of the Mellin derivative is [26]

$$\left(\beta x^{1-\alpha/\beta}D\right)^n\left[x^{r/\beta}f\left(x\right)\right]=x^{(r-n\alpha)/\beta}\sum_{k=0}^{n}S\left(n,k;\alpha,\beta,r\right)\beta^k x^k f^{(k)}\left(x\right), \tag{1.1}$$

where f is any n-times differentiable function and $S\left(n,k;\alpha,\beta,r\right)$ are a generalized Stirling number pair with three free parameters (see Section 2). Stirling numbers and their generalizations have many interesting combinatorial interpretations. These numbers are also connected with some well-known special polynomials and numbers [11, 12, 17, 23, 29–32, 35, 39, 40]. For example, the following interesting formulas for Bernoulli numbers B_n and Euler polynomials $E_n\left(x\right)$ appeared in [21]: for all $n\geq 0$,

$$B_n=\sum_{k=0}^{n}\left(-1\right)^k\frac{k!}{k+1}\left\{{n \atop k}\right\},\quad E_n\left(0\right)=\sum_{k=0}^{n}\left(-1\right)^k\frac{k!}{2^k}\left\{{n \atop k}\right\}. \tag{1.2}$$

From all these motivations, using (1.1), we introduce a new family of polynomials, namely higher order generalized geometric polynomials, and study some properties of them such as a recurrence relation, an explicit formula, and a generating function. In view of these properties, we extend the formula (1.2) for degenerate Bernoulli and degenerate Euler polynomials. We derive new explicit formulas for degenerate Bernoulli and

*Correspondence: leventkargin48@gmail.com.tr
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classical Bernoulli polynomials and then we give closed formulas for the evaluation of Carlitz's [10, Eq. (5.4)] and Howard's [24, Eq. (4.3)] sums. Finally, we calculate some integrals in terms of these polynomials. Moreover, we evaluate several power series in closed forms, one of which is the following:

$$\sum_{k=1}^{\infty}\zeta\left(k+1\right)\left(r+k\beta\mid\alpha\right)_{n}x^{k}=-\left(r\mid\alpha\right)_{n}\left(\psi\left(1-x\right)+\gamma\right)+\sum_{k=1}^{n}S\left(n,k;\alpha,\beta,r\right)k!\zeta\left(k+1,1-x\right)\left(\beta x\right)^{k},$$

where $\zeta\left(s\right)$ and $\zeta\left(s,a\right)$ are Riemann and Hurwitz zeta functions, respectively.

The summary by sections is as follows: Section 2 is the preliminary section where we give definitions and known results needed. In Section 3, we define higher order generalized geometric polynomials and study their properties. In the final section, we give several examples for the evaluation of some series and integrals.

Throughout this paper, we assume that α,β, and r are real or complex numbers.

2. Preliminaries

The generalized Stirling numbers of the first kind $S_1\left(n,k;\alpha,\beta,r\right)$ and of the second kind $S_2\left(n,k;\alpha,\beta,r\right)$ for nonnegative integer m and real or complex parameters α, β, and r, with $(\alpha,\beta,r)\neq(0,0,0)$ are defined by means of the generating function [25]

$$\left(\frac{\left(1+\beta t\right)^{\alpha/\beta}-1}{\alpha}\right)^{k}\left(1+\beta t\right)^{r/\beta}=k!\sum_{n=0}^{\infty}S_1\left(n,k;\alpha,\beta,r\right)\frac{t^n}{n!},$$

$$\left(\frac{\left(1+\alpha t\right)^{\beta/\alpha}-1}{\beta}\right)^{k}\left(1+\alpha t\right)^{-r/\alpha}=k!\sum_{n=0}^{\infty}S_2\left(n,k;\alpha,\beta,r\right)\frac{t^n}{n!},$$

with the convention $S_1\left(n,k;\alpha,\beta,r\right)=S_2\left(n,k;\alpha,\beta,r\right)=0$ when $k>n$.

As Hsu and Shiue pointed out, the definitions or generating functions generalize various Stirling-type numbers studied previously, such as:

i. $\left\{S_1\left(n,k;0,1,0\right),\ S_2\left(n,k;0,1,0\right)\right\}=\left\{s\left(n,k\right),\ S\left(n,k\right)\right\}=\left\{\left(-1\right)^{n-k}\left[\begin{smallmatrix}n\\k\end{smallmatrix}\right],\ \left\{\begin{smallmatrix}n\\k\end{smallmatrix}\right\}\right\}$

are the Stirling numbers of both kinds [13],[21, Chapter 6];

ii. $\left\{S_1\left(n,k;\alpha,1,-r\right),\ S_2\left(n,k;\alpha,1,-r\right)\right\}=\left\{\left(-1\right)^{n-k}S_1\left(n,k,r+\alpha\mid\alpha\right),\ S_2\left(n,k,r\mid\alpha\right)\right\}$

are the Howard degenerate weighted Stirling numbers of both kinds [22];

iii. $\left\{S_1\left(n,k;\alpha,1,0\right),\ S_2\left(n,k;\alpha,1,0\right)\right\}=\left\{\left(-1\right)^{n-k}S_1\left(n,k\mid\alpha\right),\ S_2\left(n,k\mid\alpha\right)\right\}$

are the Carlitz degenerate Stirling numbers of both kinds [10];

iv. $\left\{S_1\left(n,k;0,-1,r\right),\ S_2\left(n,k;0,1,r\right)\right\}=\left\{\left(-1\right)^{n-k}\left[\begin{smallmatrix}n+r\\k+r\end{smallmatrix}\right]_r,\ \left\{\begin{smallmatrix}n+r\\k+r\end{smallmatrix}\right\}_r\right\}$

are the r-Stirling numbers of both kinds [9];

v. $\{S_1(n,k;0,\beta,-1),\ S_2(n,k;0,\beta,1)\} = \{w_\beta(n,k),\ W_\beta(n,k)\}$

are the Whitney numbers of both kinds [4];

vi. $\{S_1(n,k;0,\beta,-r),\ S_2(n,k;0,\beta,r)\} = \{w_{\beta,r}(n,k),\ W_{\beta,r}(n,k)\}$

are the r-Whitney numbers of both kinds [32], and so on.

According to the generalization of Stirling numbers, Hsu and Shiue [25] defined the generalized exponential polynomials $S_n(x)$ as follows:

$$S_n(x) = \sum_{k=0}^{n} S(n,k;\alpha,\beta,r)\, x^k. \tag{2.1}$$

Later, Kargin and Corcino [26] gave an equivalent definition for $S_n(x)$ as

$$S_n(x) = \left[\left(x^{n\alpha-r}e^{-x}\right)\right]^{1/\beta}\left(\beta x^{1-\alpha/\beta}D\right)^n\left[\left(x^r e^x\right)^{1/\beta}\right], \tag{2.2}$$

and using the series

$$e^{x/\beta} = \sum_{k=0}^{\infty}\frac{x^k}{\beta^k k!},$$

in (2.2), they obtained the general Dobinksi-type formula

$$e^{x/\beta}S_n(x) = \sum_{k=0}^{\infty}\frac{(k\beta+r\mid\alpha)_n\, x^k}{\beta^k k!}. \tag{2.3}$$

Here, $(z\mid\alpha)_n$ is called the generalized factorial of z with increment α, defined by $(z\mid\alpha)_n = z(z-\alpha)\cdots(z-n\alpha+\alpha)$ for $n=1,2,\ldots,$ and $(z\mid\alpha)_0 = 1$. In particular, we have $(z\mid 1)_n = (z)_n$.

Some other properties of $S_n(x)$ can be found in [15, 16, 25, 38].

Generalized geometric polynomials $w_n(x;\alpha,\beta,r)$ are defined by means of the generalized Mellin derivative as

$$\left(\beta x^{1-\alpha/\beta}D\right)^n\left[\frac{x^{r/\beta}}{1-x}\right] = \frac{x^{(r-n\alpha)/\beta}}{1-x}w_n\left(\frac{x}{1-x};\alpha,\beta,r\right).$$

These polynomials have an explicit formula,

$$w_n(x;\alpha,\beta,r) = \sum_{k=0}^{n} S(n,k;\alpha,\beta,r)\,\beta^k k! x^k, \tag{2.4}$$

and a generating function,

$$\sum_{n=0}^{\infty} w_n(x;\alpha,\beta,r)\frac{t^n}{n!} = \frac{(1+\alpha t)^{r/\alpha}}{1-x\left((1+\alpha t)^{\beta/\alpha}-1\right)},\quad \alpha\beta\neq 0. \tag{2.5}$$

See [26] for details.

Higher order degenerate Euler polynomials are defined by means of the generating function in [10]:

$$\sum_{n=0}^{\infty} \mathcal{E}_n^{(s)}(\alpha;x)\frac{t^n}{n!} = \left(\frac{2}{(1+\alpha t)^{1/\alpha}+1}\right)^s (1+\alpha t)^{x/\alpha}. \tag{2.6}$$

From (2.6), we have

$$\lim_{\alpha\to 0} \mathcal{E}_n^{(s)}(\alpha;x) = E_n^{(s)}(x),$$

where $E_n^{(s)}(x)$ is the nth higher order Euler polynomial, which is defined by the generating function

$$\sum_{n=0}^{\infty} E_n^{(s)}(x)\frac{t^n}{n!} = \left(\frac{2}{e^t+1}\right)^s e^{xt}. \tag{2.7}$$

In some special cases,

$$\mathcal{E}_n^{(1)}(\alpha;x) = \mathcal{E}_n(\alpha;x), \ E_n^{(1)}(x) = E_n(x),$$

where $\mathcal{E}_n(\alpha;x)$ and $E_n(x)$ are the degenerate Euler and Euler polynomials, respectively.

Degenerate Bernoulli polynomials of the second kind are defined by means of the generating function in [27]:

$$\sum_{n=0}^{\infty} B_n(x\mid\alpha)\frac{t^n}{n!} = \frac{\frac{1}{\alpha}\log(1+\alpha t)}{(1+\alpha t)^{1/\alpha}-1}(1+\alpha t)^{x/\alpha}. \tag{2.8}$$

Indeed, we have

$$\lim_{\alpha\to 0} B_n(x\mid\alpha) = B_n(x),$$

where $B_n(x)$ is the nth Bernoulli polynomial, which is defined by the generating function

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{xt},$$

with $B_n(0) = B_n$ is nth Bernoulli number.

Finally, we want to mention Carlitz's degenerate Bernoulli polynomials, defined by means of the generating function in [10]:

$$\sum_{n=0}^{\infty} \beta_n(\alpha,x)\frac{t^n}{n!} = \frac{t}{(1+\alpha t)^{1/\alpha}-1}(1+\alpha t)^{x/\alpha},$$

with the relation

$$\lim_{\alpha\to 0}\beta_n(\alpha,x) = B_n(x) \text{ and } \beta_n(\alpha,0), = \beta_n(\alpha)$$

where $\beta_n(\alpha)$ is nth degenerate Bernoulli number.

3. Higher order generalized geometric polynomials

In this section, the definition of higher order generalized geometric polynomials and some properties are given. New explicit formulas for degenerate Bernoulli and Euler polynomials are also derived. Some special cases of these results are studied.

For every $s \geq 0$, taking $f(x) = 1/(1-x)^{s+1}$ in (1.1) and using

$$\frac{\partial^k}{\partial x^k} f(x) = \frac{(s+1)(s+2)\dots(s+k)}{(1-x)^{s+k+1}},$$

we have

$$\left(\beta x^{1-\alpha/\beta} D\right)^n \left[\frac{x^{r/\beta}}{(1-x)^{s+1}}\right] = \frac{x^{(r-n\alpha)/\beta}}{(1-x)^{s+1}} \sum_{k=0}^{n} S(n,k;\alpha,\beta,r) \binom{s+k}{k} k!\beta^k \left(\frac{x}{1-x}\right)^k.$$

If we define the polynomials $w_n^{(s+1)}(x;\alpha,\beta,r)$ by

$$w_n^{(s+1)}(x;\alpha,\beta,r) = \sum_{k=0}^{n} S(n,k;\alpha,\beta,r) \binom{s+k}{k} k!\beta^k x^k, \tag{3.1}$$

we reach

$$\left(\beta x^{1-\alpha/\beta} D\right)^n \left[\frac{x^{r/\beta}}{(1-x)^{s+1}}\right] = \frac{x^{(r-n\alpha)/\beta}}{(1-x)^{s+1}} w_n^{(s+1)}\left(\frac{x}{1-x};\alpha,\beta,r\right). \tag{3.2}$$

Note that if $(s,\alpha,\beta,r) = (0,\alpha,\beta,r)$, we obtain generalized geometric polynomials in [26]; if $(s,\alpha,\beta,r) = (s,0,1,0)$, we have general geometric polynomials in [5]; if $(s,\alpha,\beta,r) = (0,0,1,0)$, we obtain geometric polynomials in [5]; and if $(s,\alpha,\beta,r) = (0,0,\beta,1)$, we have Tanny–Dowling polynomials in [4]. Therefore, we call higher order generalized geometric polynomials for $w_n^{(s+1)}(x;\alpha,\beta,r)$. On the other hand, $x = 1$ and $s = 0$ in (3.1) give the numbers

$$w_n^{(1)}(1;\alpha,\beta,r) = B_n(\alpha,\beta,r) = \sum_{k=0}^{n} S(n,k;\alpha,\beta,r)\beta^k k! x^k,$$

as defined by Corcino et al. in [14]. The combinatorial interpretation and some other properties can be found in [14, 15]. Moreover, $w_n^{(s+1)}(x;\alpha,\beta,r)$ reduce to barred preferential arrangement numbers $r_{n,s}$ defined by [2, 5]

$$w_n^{(s+1)}(1;0,1,0) = r_{n,s} = \sum_{k=0}^{n} \left\{ n \atop k \right\} \binom{s+k}{k} k!,$$

which have interesting combinatorial meaning. Therefore, we may call generalized barred preferential arrangement number pair with three free parameters for

$$B_n^{(s+1)}(\alpha,\beta,r) = \sum_{k=0}^{n} S(n,k;\alpha,\beta,r) \binom{s+k}{k} k!\beta^k.$$

The combinatorial interpretation of these numbers may also be studied.

Since

$$\frac{\langle x\rangle_n}{n!} = \binom{x+n-1}{n},$$

we can write $w_n^{(s+1)}(x;\alpha,\beta,r)$ in the form

$$w_n^{(s+1)}(x;\alpha,\beta,r) = \sum_{k=0}^{n} S(n,k;\alpha,\beta,r)\langle s+1\rangle_k \beta^k x^k,$$

where $\langle x\rangle_n$ is the rising factorial defined by $\langle x\rangle_n = x(x+1)\cdots(x+n-1)$, for $n = 1,2,\ldots$, with $\langle x\rangle_0 = 1$. Furthermore, using the relation

$$\langle -x\rangle_n = (-1)^n (x)_n, \tag{3.3}$$

we can define $w_n^{(-s)}(x;\alpha,\beta,r)$ for every real $s > 0$ as

$$w_n^{(-s)}(x;\alpha,\beta,r) = \sum_{k=0}^{n} S(n,k;\alpha,\beta,r)(s)_k (-\beta)^k x^k.$$

Thus, taking $f(x) = (1-x)^s$ in (1.1), we have

$$\left(\beta x^{1-\alpha/\beta} D\right)^n \left[x^{r/\beta}(1-x)^s\right] = x^{(r-n\alpha)/\beta}(1-x)^s w_n^{(-s)}\left(\frac{x}{1-x};\alpha,\beta,r\right). \tag{3.4}$$

We turn back to $w_n^{(-s)}(x;\alpha,\beta,r)$ again in Section 4.

Now we want to deal with the properties of $w_n^{(s)}(x;\alpha,\beta,r)$, but first we give the relation between higher order generalized geometric polynomials and generalized exponential polynomials in the following theorem.

Theorem 1 *For every nonnegative integer n and $s > 0$, the generalized exponential polynomials and $w_n^{(s)}(x;\alpha,\beta,r)$ are connected by the relation*

$$w_n^{(s)}(x;\alpha,\beta,r) = \frac{1}{\Gamma(s)}\int_0^\infty z^{s-1} S_n(x\beta z) e^{-z} dz. \tag{3.5}$$

Proof Setting $x\beta z$ in (2.1), we have

$$S_n(x\beta z) = \sum_{k=0}^{n} S(n,k;\alpha,\beta,r)\beta^k x^k z^k.$$

Then multiplying both sides of the above equation with $z^{s-1}e^{-z}$, integrating it with respect to z from zero to infinity, and using the well-known identity of the gamma function

$$\frac{\Gamma(s+k)}{\Gamma(s)} = \binom{s+k-1}{k}, \quad s,k \in \mathbb{N},$$

we arrive at

$$\begin{aligned}\int_0^\infty z^{s-1}e^{-z}S_n\left(x\beta z;\alpha,\beta,r\right)dz &= \sum_{k=0}^{n} S\left(n,k;\alpha,\beta,r\right)\beta^k x^k \int_0^\infty e^{-z}z^{s+k-1}dz \\ &= \Gamma\left(s\right)\sum_{k=0}^{n} S\left(n,k;\alpha,\beta,r\right)\binom{s+k-1}{k}\beta^k x^k \\ &= \Gamma\left(s\right)w_n^{(s)}\left(x;\alpha,\beta,r\right).\end{aligned}$$

□

From Theorem 1, we can derive the properties of $w_n^{(s)}\left(x;\alpha,\beta,r\right)$ from those of $S_n\left(x\right)$. For example, if we use the extension of Spivey's Bell number formula to $S_n\left(x\right)$ in [26, 38],

$$S_{n+m}\left(x\right)=\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}S\left(m,j;\alpha,\beta,r\right)\left(j\beta-m\alpha\mid\alpha\right)_{n-k}S_k\left(x\right)x^j,$$

in (3.5), we derive a recurrence relation as

$$w_{n+m}^{(s)}\left(x;\alpha,\beta,r\right)=\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}S\left(m,j;\alpha,\beta,r\right)\left(j\beta-m\alpha\mid\alpha\right)_{n-k}\left\langle s+1\right\rangle_j\beta^j w_k^{(s+j)}\left(x;\alpha,\beta,r\right)x^j.$$

As another application of Theorem 1, we give the following theorem.

Theorem 2 *The exponential generating function for* $w_n^{(s)}\left(x;\alpha,\beta,r\right)$ *is*

$$\sum_{n=0}^{\infty}w_n^{(s)}\left(x;\alpha,\beta,r\right)\frac{t^n}{n!}=\left(\frac{1}{1-x\left(\left(1+\alpha t\right)^{\beta/\alpha}-1\right)}\right)^s\left(1+\alpha t\right)^{r/\alpha}, \tag{3.6}$$

where $\alpha\beta\neq 0$.

Proof From [25, Eq. (12)], let us write the generating function for $S_n\left(x\right)$ in the form of

$$\sum_{n=0}^{\infty}S_n\left(x\beta z\right)\frac{t^n}{n!}=\left(1+\alpha t\right)^{r/\alpha}\exp\left[xz\left(\left(1+\alpha t\right)^{\beta/\alpha}-1\right)\right].$$

Then we multiply both sides by $z^{s-1}e^{-z}$ and integrate it with respect to z from zero to infinity. In view of (3.5), we have

$$\sum_{n=0}^{\infty}w_n^{(s)}\left(x;\alpha,\beta,r\right)\frac{t^n}{n!}=\frac{\left(1+\alpha t\right)^{r/\alpha}}{\Gamma\left(s\right)}\int_0^\infty z^{s-1}e^{-z\left(1-x\left(\left(1+\alpha t\right)^{\beta/\alpha}-1\right)\right)}dz.$$

Calculating the integral on the right-hand side completes the proof. □

Setting $x=-1$ in (3.6), we have

$$w_n^{(s)}\left(-1;\alpha,\beta,r\right)=\sum_{k=0}^{n}S\left(n,k;\alpha,\beta,r\right)\binom{s+k}{k}k!\left(-\beta\right)^k=\left(r-\beta s\mid\alpha\right)_n.$$

Some other properties of $w_n^{(s)}(x;\alpha,\beta,r)$ can be derived from (3.6), but now we attend to the connection of $w_n^{(s)}(x;\alpha,\beta,r)$ with some degenerate special polynomials. If we take $x=-1/2$ in (3.6) for $\beta=1$ and compare it with (2.6), we have

$$w_n^{(s)}\left(\frac{-1}{2};\alpha,1,r\right)=\mathcal{E}_n^{(s)}(\alpha;r).$$

Thus, using (3.1) yields an explicit formula for higher order degenerate Euler polynomials in the following corollary.

Corollary 3 *For every $s\geq 0$, we have*

$$\mathcal{E}_n^{(s+1)}(\alpha;r)=\sum_{k=0}^{n}S_2(n,k,r\mid\alpha)\frac{(-1)^k\langle s+1\rangle_k}{2^k}. \tag{3.7}$$

When $s=0$, this becomes

$$\mathcal{E}_n(\alpha;r)=\sum_{k=0}^{n}S_2(n,k,r\mid\alpha)\frac{(-1)^k k!}{2^k}. \tag{3.8}$$

Now we indicate that (3.7) and (3.8) are new results. Besides, for $\alpha\to 0$, setting $x=-1/2$ in (3.6), we have [35, Theorem 3].

Secondly, if we integrate both sides of (2.5) with respect to x from -1 to 0, we have

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\int_{-1}^{0}w_n(x;\alpha,\beta,r)\,dx=\frac{\frac{\beta}{\alpha}\log(1+\alpha t)}{(1+\alpha t)^{\beta/\alpha}-1}(1+\alpha t)^{r/\alpha}. \tag{3.9}$$

In view of (2.8), for $\beta=1$, the above equation becomes

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\int_{-1}^{0}w_n(x;\alpha,1,r)\,dx=\sum_{n=0}^{\infty}B_n(r\mid\alpha)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ gives

$$\int_{-1}^{0}w_n(x;\alpha,1,r)\,dx=B_n(r\mid\alpha).$$

Finally, using (2.4) yields the following theorem.

Theorem 4 *The following equation holds for degenerate Bernoulli polynomials of the second kind:*

$$B_n(r\mid\alpha)=\sum_{k=0}^{n}S_2(n,k,r\mid\alpha)\frac{(-1)^k k!}{k+1}.$$

We note that for $\alpha \to 0$, (3.9) can be written as

$$\lim_{\alpha\to 0}\sum_{n=0}^{\infty}\frac{t^n}{n!}\int_{-1}^{0} w_n\left(x;\alpha,\beta,r\right)dx = \frac{\beta t}{e^{\beta t}-1}e^{rt}$$

$$= \sum_{n=0}^{\infty} B_n\left(\frac{r}{\beta}\right)\beta^n\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ in the above equation, we obtain

$$\lim_{\alpha\to 0}\int_{-1}^{0} w_n\left(x;\alpha,\beta,r\right)dx = B_n\left(\frac{r}{\beta}\right)\beta^n.$$

Thus, we achieve

$$B_n\left(\frac{r}{\beta}\right) = \sum_{k=0}^{n} W_{\beta,r}\left(n,k\right)\frac{\left(-1\right)^k k!}{\beta^{n-k}\left(k+1\right)},$$

which was also given in [35] with a different proof.

The next result is on the explicit formulas for Carlitz's degenerate Bernoulli polynomials.

Theorem 5 *For every $s \geq 0$, we have*

$$\beta_{n+1}\left(\alpha,r\right) - \beta_{n+1}\left(\alpha,r-s\right) = \left(n+1\right)\sum_{k=0}^{n} S_2\left(n,k,r \mid \alpha\right)\frac{\left(-1\right)^k \langle s\rangle_{k+1}}{k+1}. \tag{3.10}$$

When $s = r$ and $s = \alpha$, this becomes

$$\beta_{n+1}\left(\alpha,r\right) = \beta_{n+1}\left(\alpha\right) + \left(n+1\right)\sum_{k=0}^{n} S_2\left(n,k,r \mid \alpha\right)\frac{\left(-1\right)^k \langle r\rangle_{k+1}}{k+1} \tag{3.11}$$

and

$$\beta_n\left(\alpha,r-\alpha\right) = \sum_{k=0}^{n} S_2\left(n,k,r \mid \alpha\right)\frac{\left(-1\right)^k \langle \alpha+1\rangle_{k}}{k+1}, \tag{3.12}$$

respectively.

Proof If we integrate both sides of (3.6) with respect to x from -1 to 0, we have

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\int_{-1}^{0} w_n^{(s+1)}\left(x;\alpha,\beta,r\right)dx = \frac{1}{st}\left[\frac{t\left(1+\alpha t\right)^{r/\alpha}}{\left(1+\alpha t\right)^{\beta/\alpha}-1} - \frac{t\left(1+\alpha t\right)^{(r-\beta s)/\alpha}}{\left(1+\alpha t\right)^{\beta/\alpha}-1}\right]. \tag{3.13}$$

For $\beta = 1$, the above equation becomes

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\int_{-1}^{0} w_n^{(s+1)}\left(x;\alpha,1,r\right)dx = \frac{1}{s}\left[\sum_{n=0}^{\infty}\frac{\beta_{n+1}\left(\alpha,r\right)-\beta_{n+1}\left(\alpha,r-s\right)}{n+1}\right]\frac{t^n}{n!}.$$

Equating the coefficients of $\frac{t^n}{n!}$ in the above equation, we obtain

$$\int_{-1}^{0} w_n^{(s+1)}(x;\alpha,1,r)\,dx = \frac{\beta_{n+1}(\alpha,r)-\beta_{n+1}(\alpha,r-s)}{s(n+1)}.$$

Finally, using (3.1) yields (3.10).

Setting $s=\alpha$ in (3.10) and using the identity [10, Eq. (5.10)]

$$\beta_n(\alpha,z+\alpha)=\beta_n(\alpha,z)+\alpha n\beta_{n-1}(\alpha,z),$$

we get (3.12). □

Let us return to (3.13) again. For $\alpha\to 0$, (3.13) can be written as

$$\lim_{\alpha\to 0}\sum_{n=0}^{\infty}\frac{t^n}{n!}\int_{-1}^{0} w_n^{(s+1)}(x;\alpha,\beta,r)\,dx = \frac{1}{st}\left[\frac{te^{rt}}{e^{\beta t}-1}-\frac{te^{(r-\beta s)t}}{e^{\beta t}-1}\right]$$
$$=\frac{1}{s}\left[\sum_{n=0}^{\infty}\frac{B_{n+1}\left(\frac{r}{\beta}\right)-B_{n+1}\left(\frac{r}{\beta}-s\right)}{n+1}\beta^{n+1}\right]\frac{t^n}{n!}.$$

Since $S_2(n,k;0,\beta,r)=W_{\beta,r}(n,k)$, we derive the values of Bernoulli polynomials at rational arguments in the following theorem.

Theorem 6 *For every $s\geq 0$ and $\beta\neq 0$, we have*

$$B_{n+1}\left(\frac{r}{\beta}\right)-B_{n+1}\left(\frac{r}{\beta}-s\right)=(n+1)\sum_{k=0}^{n}W_{\beta,r}(n,k)\frac{(-1)^k\langle s\rangle_{k+1}}{\beta^{n+1-k}(k+1)}. \tag{3.14}$$

For $s=r/\beta$, this becomes

$$B_{n+1}\left(\frac{r}{\beta}\right)=B_{n+1}+(n+1)\sum_{k=0}^{n}W_{\beta,r}(n,k)\frac{(-1)^k\langle r\mid\beta\rangle_{k+1}}{\beta^{n+2}(k+1)}, \tag{3.15}$$

where $\langle x\mid\alpha\rangle_n=x(x+\alpha)\cdots(x+(n-1)\alpha)$, for $n=1,2,\ldots$, with $\langle x\mid\alpha\rangle_0=1$.

Note that setting $\beta=1$ and $r=0$ in (3.14), replacing $-s$ with x in that equation, and using (3.3) gives the well-known identity for Bernoulli polynomials [37, Eq. (15.39)]:

$$B_{n+1}(x)=B_{n+1}+\sum_{k=0}^{n}\frac{(n+1)}{k+1}\left\{{n\atop k}\right\}(x)_{k+1}.$$

On the other hand, taking $s=r$ in (3.15) gives a different representation of the above equation as

$$B_{n+1}(r)=B_{n+1}+\sum_{k=0}^{n}\frac{(-1)^k(n+1)}{k+1}\left\{{n+r\atop k+r}\right\}_r\langle r\rangle_{k+1}.$$

For the consequences of Theorem 5 and Theorem 6, we deal with Carlitz's identity [10, Eq. (5.4)]:

$$\sum_{j=0}^{r-1}(j \mid \alpha)_m = \frac{1}{m+1}\left[\beta_{m+1}(\alpha, r) - \beta_{m+1}(\alpha)\right],$$

and Howard's identity [24, Eq. (4.3)]:

$$\sum_{j=0}^{m-1}(r+\beta j)^n = \frac{\beta^n}{n+1}\left[B_{n+1}\left(m+\frac{r}{\beta}\right) - B_{n+1}\left(\frac{r}{\beta}\right)\right],$$

respectively. From (3.11), we reach a new representation for Carlitz's identity. On the other hand, replacing $-s$ with m in (3.14) and then using (3.3) gives a new representation for Howard's identity, as in the following corollary.

Corollary 7 *For every integers > 0, we have*

$$\sum_{j=0}^{r-1}(j \mid \alpha)_n = \sum_{k=0}^{n} S_2(n, k, r \mid \alpha)\frac{(-1)^k \langle r\rangle_{k+1}}{k+1}.$$

For n and m that are nonnegative integers with $m > 0$ and $\beta \neq 0$, we have

$$\sum_{j=0}^{m-1}(r+\beta j)^n = \sum_{k=0}^{n} W_{\beta,r}(n, k)\frac{\beta^{k-1}(m)_{k+1}}{k+1}.$$

4. Some examples of series and integrals evaluation

In this section, we extend some results given in [5, 8, 26, 34] and present additional examples.

The following two examples are on the generalized exponential polynomials. Consider the function

$$f(x) = \cosh(x/\beta) = \sum_{k=0}^{\infty}\frac{x^{2k}}{\beta^{2k}(2k)!}.$$

From (1.1), we have

$$\begin{aligned}\left(\beta x^{1-\alpha/\beta}D\right)^n\left[x^{r/\beta}\left(e^{x/\beta}+e^{-x/\beta}\right)\right] &= 2\sum_{k=0}^{\infty}\frac{1}{\beta^{2k}(2k)!}\left(\beta x^{1-\alpha/\beta}D\right)^n\left[x^{(2k\beta+r)/\beta}\right]\\ &= 2x^{(r-n\alpha)/\beta}\sum_{k=0}^{\infty}\frac{(2k\beta+r \mid \alpha)_n}{\beta^{2k}(2k)!}x^{2k}.\end{aligned}$$

Using (2.2), the left-hand side of the above equation can be written as

$$\begin{aligned}&\left(\beta x^{1-\alpha/\beta}D\right)^n\left[x^{r/\beta}\left(e^{x/\beta}+e^{-x/\beta}\right)\right]\\ &= \left(\beta x^{1-\alpha/\beta}D\right)^n\left[x^{r/\beta}e^{x/\beta}\right] + \left(\beta x^{1-\alpha/\beta}D\right)^n\left[x^{r/\beta}e^{-x/\beta}\right]\\ &= x^{(r-n\alpha)/\beta}e^{x/\beta}S_n(x) + x^{(r-n\alpha)/\beta}e^{-x/\beta}S_n(-x).\end{aligned}$$

Combining these two identities gives

$$2\sum_{k=0}^{\infty}\frac{(2k\beta+r\mid\alpha)_n}{\beta^{2k}(2k)!}x^{2k}=e^{x/\beta}S_n(x)+e^{-x/\beta}S_n(-x).$$

Similarly, taking $f(x)=\sinh x$ in (1.1) gives

$$2\sum_{k=0}^{\infty}\frac{((2k+1)\beta+r\mid\alpha)_n}{\beta^{2k+1}(2k+1)!}x^{2k+1}=e^{x/\beta}S_n(x)-e^{-x/\beta}S_n(-x)$$

Finally, setting $x=2\pi i\beta$ in the above equations yields our first example.

Example 8 *We have*

$$\sum_{k=0}^{\infty}(2k\beta+r\mid\alpha)_n\frac{(-1)^k(2\pi)^{2k}}{(2k)!}=\sum_{j=1}^{\lfloor n/2\rfloor}S(n,2j;\alpha,\beta,r)(-1)^j(2\pi\beta)^{2j},\tag{4.1}$$

and

$$\sum_{k=0}^{\infty}((2k+1)\beta+r\mid\alpha)_n\frac{(-1)^k(2\pi)^{2k}}{(2k+1)!}=\beta\sum_{j=1}^{\lfloor n/2\rfloor}S(n,2j+1;\alpha,\beta,r)(-1)^j(2\pi\beta)^{2j}.\tag{4.2}$$

Note that for $\alpha=0$, (4.1) and (4.2) reduce to [34, p. 403].

Example 9 *In [7, Eq. (8.4)], we have*

$$\sum_{k=1}^{\infty}\frac{x^k}{\beta^k k!}\left[1^p+2^p+\ldots+k^p\right]=e^{x/\beta}\sum_{j=1}^{p}\left\{{p+1\atop j}\right\}\frac{x^j}{j\beta^j}.$$

Multiplying both sides of the above equation by $x^{r/\beta}$ and applying the operator $\left(\beta x^{1-\alpha/\beta}D\right)^n$, we obtain

$$\sum_{k=1}^{\infty}\frac{\left(\beta x^{1-\alpha/\beta}D\right)^n\left[x^{k+r/\beta}\right]}{\beta^k k!}\left[1^p+2^p+\ldots+k^p\right]=\sum_{j=1}^{p}\left\{{p+1\atop j}\right\}\frac{\left(\beta x^{1-\alpha/\beta}D\right)^n\left[x^{j+r/\beta}e^{x/\beta}\right]}{j\beta^j}.$$

From (2.2), the right-hand side becomes

$$\sum_{j=1}^{p}\left\{{p+1\atop j}\right\}\frac{\left(\beta x^{1-\alpha/\beta}D\right)^n\left[x^{j+r/\beta}e^{x/\beta}\right]}{j\beta^j}=x^{(r-n\alpha)/\beta}e^{x/\beta}\sum_{j=1}^{p}\left\{{p+1\atop j}\right\}S_n(x;\alpha,\beta,r+j)\frac{x^{j/\beta}}{j\beta^j}.$$

Then we have the following closed formula:

$$\sum_{k=1}^{\infty}\frac{(r+k\beta\mid\alpha)_n x^k}{\beta^k k!}\left[1^p+2^p+\ldots+k^p\right]=e^{x/\beta}\sum_{j=1}^{p}\left\{{p+1\atop j}\right\}S_n(x;\alpha,\beta,r+j)\frac{x^{j/\beta}}{j\beta^j}.$$

For $(\alpha,\beta,r)=(0,1,0)$, the above result becomes

$$\sum_{k=1}^{\infty}\frac{k^n x^k}{k!}\left[1^p+2^p+\ldots+k^p\right]=e^x\sum_{j=1}^{p}\left\{{p+1\atop j}\right\}B_{n,j}(x)\frac{x^j}{j},$$

where $B_{n,r}(x)$ is the nth r-Bell polynomial in [33].

In the following proposition, we extend the following series [26]:

$$\sum_{k=0}^{\infty}(r+k\beta\mid\alpha)_n\,x^k=\frac{1}{(1-x)}w_n\left(\frac{x}{1-x};\alpha,\beta,r\right), \tag{4.3}$$

and [8, Eq. 9]

$$\sum_{k=0}^{\infty}\binom{s}{k}k^n x^k=(1+x)^s\,w_n^{(-s)}\left(\frac{-x}{1+x}\right).$$

Proposition 10 *For any nonnegative integers n and $s>0$, we have*

$$\sum_{k=0}^{\infty}\binom{s+k}{k}(r+k\beta\mid\alpha)_n\,x^k=\frac{1}{(1-x)^{s+1}}w_n^{(s+1)}\left(\frac{x}{1-x};\alpha,\beta,r\right), \tag{4.4}$$

and

$$\sum_{k=0}^{\infty}\binom{s}{k}(r+k\beta\mid\alpha)_n\,x^k=(1+x)^s\,w_n^{(-s)}\left(\frac{-x}{1+x};\alpha,\beta,r\right). \tag{4.5}$$

Proof Let us apply (1.1) to both sides of the function

$$\frac{1}{(1-x)^{s+1}}=\sum_{k=0}^{\infty}\binom{s+k}{k}x^k.$$

From (3.2), we obtain

$$\begin{aligned}\frac{x^{(r-n\alpha)/\beta}}{(1-x)^{s+1}}w_n^{(s+1)}\left(\frac{x}{1-x};\alpha,\beta,r\right)&=\sum_{k=0}^{\infty}\binom{s+k}{k}\left(\beta x^{1-\alpha/\beta}D\right)^n\left[x^{k+r/\beta}\right]\\&=x^{(r-n\alpha)/\beta}\sum_{k=0}^{\infty}\binom{s+k}{k}(r+k\beta\mid\alpha)_n\,x^k.\end{aligned}$$

Then, simplifying $x^{(r-n\alpha)/\beta}$ on both side yields the desired equation.

For the proof of (4.5), for every $s>0$, we take

$$f(x)=(1+x)^s=\sum_{k=0}^{\infty}\binom{s}{k}x^k$$

in (1.1) and use (3.4). □

The next result is on a closed form for the evaluation of a power series that has the coefficients including the Riemann zeta function.

Theorem 11 *For $|x|<1$, we give*

$$\sum_{k=1}^{\infty}\zeta(k+1)(r+k\beta\mid\alpha)_n\,x^k=-(r\mid\alpha)_n(\psi(1-x)+\gamma)+\sum_{k=1}^{n}S(n,k;\alpha,\beta,r)\,k!\zeta(k+1,1-x)(\beta x)^k. \tag{4.6}$$

As a consequence of Theorem 11, the following sums are obtained:
For $(\alpha,\beta,r)=(0,\beta,r)$,

$$\sum_{k=1}^{\infty}\zeta(k+1)(r+k\beta)^n x^k=-r^n(\psi(1-x)+\gamma)+\sum_{k=1}^{n}W_{\beta,r}(n,k)\,k!\zeta(k+1,1-x)(\beta x)^k. \tag{4.7}$$

For $(\alpha,\beta,r)=(0,1,r)$,

$$\sum_{k=1}^{\infty}\zeta(k+1)(r+k)^n x^k=-r^n(\psi(1-x)+\gamma)+\sum_{k=1}^{n}\left\{\begin{matrix}n+r\\k+r\end{matrix}\right\}_r k!\zeta(k+1,1-x)x^k.$$

For $(\alpha,\beta,r)=(0,1,1)$, using the relation $\left\{\begin{matrix}n\\k\end{matrix}\right\}_0=\left\{\begin{matrix}n\\k\end{matrix}\right\}_1=\left\{\begin{matrix}n\\k\end{matrix}\right\}$,

$$\sum_{k=1}^{\infty}\zeta(k+1)(k+1)^n x^k=-(\psi(1-x)+\gamma)+\sum_{k=1}^{n}\left\{\begin{matrix}n+1\\k+1\end{matrix}\right\}k!\zeta(k+1,1-x)x^k. \tag{4.8}$$

Note that for $x=-1/2$, (4.7) reduces to [34]. Besides, (4.8) is a special case of [8, Proposition 20].

Proof [Proof of Theorem 11]Let us take $f(x)=\psi(x)$ in (1.1), where $\psi(x)$ is the digamma function, which can be given by [1, Eq. 6.3.14]

$$\psi(x+1)=-\gamma+\sum_{k=1}^{\infty}\zeta(k+1)(-1)^{k+1}x^k,\quad |x|<1. \tag{4.9}$$

Here, γ is Euler's constant. Then we have

$$\left(\beta x^{1-\alpha/\beta}D\right)^n\left[x^{r/\beta}\psi(x+1)\right]=x^{(r-n\alpha)/\beta}\sum_{k=0}^{n}S(n,k;\alpha,\beta,r)\,\beta^k x^k\psi^{(k)}(x+1), \tag{4.10}$$

where $\psi^{(m)}(x)$ is the polygamma function, and it can be written more compactly as [1, Eq. 6.4.10]

$$\psi^{(m)}(x)=(-1)^{m+1}m!\zeta(m+1,x). \tag{4.11}$$

Here $m>0$, and x is any complex number not equal to a negative integer. Using (4.9) and (4.11) in (4.10), we have the desired equation. □

Now we want to add some examples for the evaluation of integrals. Before giving the examples, we need to mention that we use the well-known estimate for the gamma function in the rest of this section:

$$|\Gamma(x+iy)|\sim\sqrt{2\pi}\,|y|^{x-\frac{1}{2}}e^{-x-\frac{\pi}{2}|y|},$$

$(|y|\to\infty)$ for any fixed real x. This explains the behavior of the gamma function on vertical lines $\{t=a+iz:-\infty<z<\infty,\ 0<a<1\}$ in [8].

Theorem 12 *For every $s\geq 0$, every $0<x<1$, and $0<a<1$, we have*

$$\frac{1}{(1+x)^{s+1}}w_n^{(s+1)}\left(\frac{-x}{1+x};\alpha,\beta,r\right)=\frac{1}{2\pi i\Gamma(s+1)}\int_{a-i\infty}^{a+i\infty}(r-\beta t\mid\alpha)_n\,x^{-t}\Gamma(t)\Gamma(s+1-t)\,dt. \tag{4.12}$$

Besides, for all $x>0,\ \operatorname{Re}(\beta)>0,\ \operatorname{Re}(r)>0,\ n=0,1,2,\ldots,$ *and* $a>n+1,$

$$\frac{e^{-rx}}{1-e^{-\beta x}}w_n\left(\frac{e^{-\beta x}}{1-e^{-\beta x}};\alpha,\beta,r\right)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}x^{-t}\sum_{k=0}^{\infty}\frac{(k\beta+r\mid\alpha)_n}{(k\beta+r)^t}\Gamma(t)\,dt. \tag{4.13}$$

The interesting part of (4.13) appears when $\alpha=0$ as

$$\frac{e^{-rx}}{1-e^{-\beta x}}w_n\left(\frac{e^{-\beta x}}{1-e^{-\beta x}};0,\beta,r\right)=\frac{\beta^n}{2\pi i}\int_{a-i\infty}^{a+i\infty}(\beta x)^{-t}\,\zeta\left(t-n,\frac{r}{\beta}\right)\Gamma(t)\,dt, \tag{4.14}$$

where $w_n(x;0,\beta,r)=\sum_{k=0}^{n}W_{\beta,r}(n,k)\,\beta^k k!x^k.$

For $\beta=1$ and $n=0$ in (4.14), we have the well-known inverse Mellin transformation of the Hurwitz zeta function in [3, Theorem 12.2]. Moreover, (4.12) and (4.14) are the generalization of identities in [8, Proposition 16].

Proof [Proof of Theorem 12]Let us start from [36, Formula 5.37]

$$\frac{1}{(1+x)^{s+1}}=\frac{1}{2\pi i\Gamma(s+1)}\int_{a-i\infty}^{a+i\infty}x^{-t}\Gamma(t)\,\Gamma(s+1-t)\,dt,$$

where $s\geq 0,\ 0<x<1.$ Apply (1.1) to both sides of the above integral to obtain

$$\left(\beta x^{1-\alpha/\beta}D\right)^n\left[\frac{x^{r/\beta}}{(1+x)^{s+1}}\right]=\frac{x^{(r-n\alpha)/\beta}}{2\pi i\Gamma(s+1)}\int_{a-i\infty}^{a+i\infty}(r-\beta t\mid\alpha)_n\,x^{-t}\Gamma(t)\,\Gamma(s+1-t)\,dt.$$

For the left-hand side of the above equation, we derive

$$\begin{aligned}\left(\beta x^{1-\alpha/\beta}D\right)^n\left[\frac{x^{r/\beta}}{(1+x)^{s+1}}\right]&=\sum_{k=0}^{\infty}\binom{s+k}{k}(-1)^k\left(\beta x^{1-\alpha/\beta}D\right)^n x^{(k\beta+r)/\beta}\\&=x^{(r-n\alpha)/\beta}\sum_{k=0}^{\infty}\binom{s+k}{k}(r+k\beta\mid\alpha)_n(-x)^k\\&=\frac{x^{(r-n\alpha)/\beta}}{(1+x)^{s+1}}w_n^{(s+1)}\left(\frac{-x}{1+x};\alpha,\beta,r\right).\end{aligned}$$

Therefore, we have (4.12).

Replacing x by $(r+k\beta)\,x$, multiplying both sides by $(r+k\beta\mid\alpha)_n$, and summing for $k=0,1,\ldots,$ in the integral

$$e^{-x}=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}x^{-t}\Gamma(t)\,dt,$$

we have

$$e^{-rx}\sum_{k=0}^{\infty}(r+k\beta\mid\alpha)_n\left(e^{-\beta x}\right)^k=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}x^{-t}\sum_{k=0}^{\infty}\frac{(k\beta+r\mid\alpha)_n}{(k\beta+r)^t}\Gamma(t)\,dt.$$

Using (4.3) in the above integral gives (4.13). □

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