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# Some Integral Formulas on Compact Riemannian Submanifolds in $R^m$

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**Abstract:**

In this paper, integral formulas are given for Riemannian manifolds of constant curvature.

**Keywords:** Riemannian manifold, compact manifold.

# $R^m$ de Kompakt Riemann Alt manifoldları Üzerinde Bazı İntegral Formülleri

**Özet:**

Bu makalede sabit eğrilikli Riemann manifoldları için integral formülleri verildi.

**Anahtar Kelimeler:** Riemann manifoldu, kompakt manifold.

## 1. Introduction

Cao [1] obtained integral formulas for minimal space-like hypersurfaces in (n+1)-dimensional indefinite space form. Ximin [2] obtained integral formula for minimal space-like hypersurfaces in (n+p)-dimensional indefinite space form. Similar integral formulas were obtained in Sasakian space form by Bektaş and Turhan [3]. In this paper we are going to obtain these integral formulas for Riemannian manifold of constant curvature as following.

**Theorem 1.** Let $M^n$ be an n-dimensional compact submanifold of an m-dimensional Riemannian manifold of $R^m$, then

$$\int_{M^n} \left\{ \frac{1}{2} \sum K_{mijk}^2 + \sum R_{mj}^2 - nKS \right\} dV \le 0 \tag{1}$$

where $\sum K_{mijk}^2$ is the square length of the sectional curvature, $\sum R_{mj}^2$ is the square length of the Ricci curvature tensor, S is the scalar curvature, dV is the volume element of $M^n$

**Theorem 2.** Let $M^n$ be n-dimensional compact submanifold of m-dimensional Riemannian manifold of $R^m$ with constant sectional curvature, then

$$\int_{M^n} \left\{ \frac{1}{2} \sum K_{mijk}^2 + \sum R_{mj}^2 - nKS \right\} dV \le n^2 K(n-1)(K+\alpha^2) Vol(M^n) \tag{2}$$

where $\sum K^2_{mijk}$ is the square length of the sectional curvature, $\sum R^2_{mj}$ is the square length of the Ricci curvature tensor, S is the scalar curvature, dV the volume element of $M^n$, $\alpha$ is the mean curvature of $M^n$.

**Theorem 3.** Let $M^n$ be n-dimensional compact submanifold of an m-dimensional Riemannian manifold of $R^m$, then

$$\int_{M^n}\left\{\frac{1}{2}\sum K^2_{mijk}-\left(n^2(n+2)+n-\frac{1}{n}-2\right)h^2+(n^3-3n^2+2)Kh\right\}dV \leq nK^2(n-1)Vol(M^n) \tag{3}$$

where $\sum K^2_{mijk}$ is the square length of the sectional curvature, dV is the volume element of $M^n$.

## 2. Local Formulas

We will use the same notation and terminologies as in [4] unless otherwise stated.

Let $M^n$ be an oriented Riemannian manifold of dimension $m \geq 3$, i.e., there is given in each tangent space $T_p(R^m)$, $p \in R^m$, a positive definite scalar product which varies in a $C^\infty$ manner with p. The scalar product will be denoted by $\langle \xi, \eta \rangle$, $\xi$, $\eta \in T_p(R^m)$. We will agree on the following ranges of indices unless otherwise stated:

$$1 \leq i, j, k, \ldots \leq n \quad ; \quad n+1 \leq r, s, t, \ldots \leq m \quad ; \quad 1 \leq A, B, C, \ldots \leq m \tag{4}$$

By an orthonormal frame $(p, e_1, e_2, \ldots, e_m)$ is meant an ordered set of m vectors in the same tangent space $T_p(R^m)$, which defines the orientation of $R^m$ and which satisfies the relation $\langle e_A, e_B \rangle = \delta_{AB}$, where $\delta_{AB}$ are the Kronecker deltas. The frame $e_A$ defines uniqely a dual coframe $\overline{w}_B$ in the cotangent space $T_p^*(R^m)$ and vice versa. The element of arc is given by

$$ds^2 = \sum_A \overline{w}_A^2 \ . \tag{5}$$

Let $F(R^m)$ be the bundle of orthonormal frames of $R^m$. The structure equations of $F(R^m)$ are given by

$$d\overline{w}_A = \sum \overline{w}_{AB} \wedge \overline{w}_B \ ,$$

$$d\overline{w}_{AB} = \sum \overline{w}_{AC} \wedge \overline{w}_{CB} + \overline{\Omega}_{AB} \quad , \quad \overline{\Omega}_{AB} = \frac{1}{2}\sum \overline{R}_{ABCD}\overline{w}_C \wedge \overline{w}_D , \tag{6}$$

$$\overline{w}_{AB} + \overline{w}_{BA} = 0,$$

where $\overline{w}_{AB}$ and $\overline{\Omega}_{AB}$ are the connection foms and the curvature forms, respectively.

Let $x : M^n \to R^m$ be an isometric immersion of an n-dimensional Riemannian manifold $M^n$ into $R^m$. Let $F(M^n)$ be the bundle of orthonormal frames of $M^n$, and B the set of elements $b = (p, e_1, ..., e_m)$ such that $(p, e_1, ..., e_n) \in F(M^n)$ and $(x(p), e_1, ..., e_m) \in F(R^m)$. Let $\tilde{x} : B \to F(R^m)$ be defined by $\tilde{x}(p, e_1, ..., e_m) = (x(p), e_1, ..., e_m)$, and $w_A, w_{AB}, \Omega_{AB}$ be the induced forms on B from $\overline{w}_A, \overline{w}_{AB}, \overline{\Omega}_{AB}$ by the mapping $\tilde{x}$. Then we have

$$w_r = 0. \tag{7}$$

Taking its exterior derivative and making use of (6), we get

$$\sum w_i \wedge w_{ir} = 0 \tag{8}$$

By Cartan's lemma we have

$$w_{ir} = \sum A_{rij} w_j \quad , \quad A_{rij} = A_{rji} \ . \tag{9}$$

The mean curvature vector is defined by

$$H = \frac{1}{n} \sum_{i,r} A_{rii} e_r \ . \tag{10}$$

Let $M^n$ be a pseudo-umbilical submanifold of $R^m$, and we always choose the first unit normal vector $e_{n+1}$ in the direction of mean curvature vector, i.e.

$$H = \alpha e_{n+1} \quad , \quad \alpha > 0 \tag{11}$$

where $\alpha$ is the mean curvature of $M^n$ in $R^m$. By the definition of pseudo-umbilical submanifold, we get

$$A_{n+1,ij} = \alpha \delta_{ij} \ , \tag{12}$$

and

$$\sum_i A_{rii} = 0 \quad , \quad r = n+2, ..., m \ . \tag{13}$$

Now, assume that $R^m$ has constant sectional curvature K, i.e.

$$\overline{\Omega}_{AB} = -K\overline{w}_A \wedge \overline{w}_B . \tag{14}$$

Then by (6) and (9), we have

$$dw_{ij} = \sum w_{ik} \wedge w_{kj} - \sum w_{ir} \wedge w_{jr} - Kw_i \wedge w_j . \tag{15}$$

Then the curvature forms $\Omega_{ij}$ of $M^n$ can be written as

$$\Omega_{ij} = dw_{ij} - \sum w_{ik} \wedge w_{kj} = -Kw_i \wedge w_j - \sum w_{ir} \wedge w_{jr} . \tag{16}$$

By expressing $\Omega_{ij}$ as a 2-form of $w_k$:

$$\Omega_{ij} = \frac{1}{2}\sum K_{ijkl} w_k \wedge w_l ,$$

we obtain

$$K_{ijkl} = -K(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum (A_{rik}A_{rjl} - A_{ril}A_{rjk}) . \tag{17}$$

Thus the Ricci tensor $R_{jk}$ is given by

$$R_{jk} \equiv \sum K_{ijki} = (n-1)K\delta_{jk} - \sum A_{rik}A_{rji} + \sum A_{rii}A_{rjk} \tag{18}$$

Thus, by (12) and (18), we get

$$R_{jk} = (n-1)(K+\alpha^2)\delta_{jk} - \sum_{r=n+2}^{m}\sum_{i=1}^{n} A_{rik}A_{rji} . \tag{19}$$

Let $z = \sum z_i e_i$ be a unit tangent vector field over $M^n$. If, for every normal vector field $u = \sum_{r=n+2}^{m} u_r e_r$ perpendicular to the mean curvature vector field H we have

$$\langle z, Du \rangle = 0 \quad , \tag{20}$$

then z is called a principal direction in $R^m$. The Ricci curvature R(z) in the direction z is defined by

$$R(z) = \sum R_{jk} z_j z_k \ . \tag{21}$$

Let $x : M^n \to R^m$ be an immersion of $M^n$ in a Riemannian manifold of constant sectional curvature K as a pseudo-umbilical submanifold, the scalar curvature S and the mean curvature $\alpha$ satisfy the inequality:

$$S \equiv \sum R_{jj} \le n(n-1)(K + \alpha^2) . \tag{22}$$

If the dimension of $M^n$ is $\ge 3$, then the equality sign of (22) holds when and only when $M^n$ has constant sectional curvature $K + \alpha^2$.

**3. Proof of the theorems**

**Proof of the Theorem1.** From the definition of Laplacian we have

$$\Delta A_{rij} = \sum A_{rim} K_{mkjk} + \sum A_{rkm} K_{mijk} \ . \tag{23}$$

From (13), (17) and (23), we get

$$\sum A_{rij} \Delta A_{rij} = \sum A_{rij} A_{rmk} K_{mijk} + \sum A_{rij} A_{rim} K_{mkjk}$$

$$\begin{aligned} \sum A_{rij} \Delta A_{rij} &= \frac{1}{2} \sum (A_{rij} A_{rmk} - A_{rmj} A_{rik}) K_{mijk} \\ &\quad + \sum (A_{rij} A_{rim} - A_{rii} A_{rjm}) (-R_{mj}) \end{aligned} \tag{24}$$

If equality (17) is used in the first term at the right side of the equality (24), we have

$$\frac{1}{2} \sum (A_{rmk} A_{rij} - A_{rmj} A_{rik}) = \frac{1}{2} \left[ - K (\delta_{mk} \delta_{ij} - \delta_{mj} \delta_{ik}) + K_{mijk} \right] \ . \tag{25}$$

If the equality (17) is used in second term at the right side of equality (24), we find

$$\sum (A_{rim} A_{rji} - A_{rii} A_{rjm}) = -K (\delta_{im} \delta_{ji} - \delta_{ii} \delta_{jm}) + K_{ijim} \ . \tag{26}$$

If (25) and (26) are written in (24), we have

$$\begin{aligned} \sum A_{rij} \Delta A_{rij} &= \frac{1}{2} \sum \left[ - K (\delta_{mk} \delta_{ij} - \delta_{mj} \delta_{ik}) + K_{mijk} \right] K_{mijk} \\ &\quad + \sum \left[ - K (\delta_{im} \delta_{ji} - \delta_{ii} \delta_{jm}) + K_{ijim} \right] (- R_{mj}) \end{aligned}$$

or

$$\sum A_{rij}\,\Delta A_{rij} = \frac{1}{2}\sum K^2_{mijk} + \sum R^2_{mj} + \frac{1}{2}\sum\left[-K\left(\delta_{mk}\delta_{ij} - \delta_{mj}\delta_{ik}\right)\right]K_{mijk} + \sum\left[K\left(\delta_{im}\delta_{ji} - \delta_{ii}\delta_{jm}\right)\right]\left(R_{mj}\right). \tag{27}$$

After some calculation last two terms at the right side of (27) are obtained as the following:

$$\frac{1}{2}\sum\left[-K\left(\delta_{mk}\delta_{ij} - \delta_{mj}\delta_{ik}\right)\right]K_{mijk} + \sum\left[K\left(\delta_{im}\delta_{ji} - \delta_{ii}\delta_{jm}\right)\right]\left(R_{mj}\right) = -nKS\ . \tag{28}$$

If (28) is written in (27), we find

$$\sum A_{rij}\,\Delta A_{rij} = \frac{1}{2}\sum K^2_{mijk} + \sum R^2_{mj} - nKS \tag{29}$$

From Cao [2] and Ximin [3], we have the following:

$$\int_{M^n}\left\{\frac{1}{2}\sum K^2_{mijk} + \sum R^2_{mj} - nKS\right\}dV \le 0\,.$$

This completes proof of Theorem1.

**Proof of the Theorem2.** If (22) is considered in (1), the proof of the theorem is trivial.

In order to prove Theorem3, we need the following, we have lemma.

**Lemma 3.2.** Let $a_1, \ldots, a_n$ be real numbers, then

$$\sum (a_i)^2 \ge \frac{1}{n}\left(\sum a_i\right)^2\ . \tag{30}$$

where the equality sign holds when and only when $a_1 = \ldots = a_n$.

**Proof of the Theorem3.** If (12) is considered in (19), we have

$$R_{mj} = (n-1)(K+\alpha^2)\delta_{mj} - \sum \alpha^2\delta_{ij}\delta_{mi}\,. \tag{31}$$

If (30) is used in (31), we find

$$\sum R^2_{mj} = n(n-1)^2(K+\alpha^2)^2 - 2(n-1)(K+\alpha^2)\alpha^2 + \sum \alpha^4$$

$$\ge n(n-1)^2(K+\alpha^2)^2 - 2(n-1)(K+\alpha^2)\alpha^2 + \frac{1}{n}\sum(\alpha^2)^2\,.$$

If we put $\sum \alpha^2 = h$, we get

$$\sum R_{mj}^2 \geq n(n-1)^2(K+\alpha^2)^2 - 2(n-1)(K+\alpha^2)\alpha^2 + \frac{1}{n}h^2 \tag{32}$$

If (32) and (22) are used in (1), we obtain

$$\int_{M^n} \left\{ \frac{1}{2}\sum K_{mijk}^2 - \left( n^2(n+2) + n - \frac{1}{n} - 2 \right) h^2 + (n^3 - 3n^2 + 2)Kh - nK^2(n-1) \right\} dV \leq 0$$

or

$$\int_{M^n} \left\{ \frac{1}{2}\sum K_{mijk}^2 - \left( n^2(n+2) + n - \frac{1}{n} - 2 \right) h^2 + (n^3 - 3n^2 + 2)Kh \right\} dV \leq nK^2(n-1)Vol(M^n).$$

This completes the proof.

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