

On the fine spectra of the Jacobi matrices on c_0, c, ℓ_p ($1 \leq p \leq \infty$) and bv_p ($1 \leq p < \infty$)

Nuh DURNA^{1,*} , Mustafa YILDIRIM¹ , Abbas KILIÇ¹ 

¹Sivas Cumhuriyet University, Department of Mathematics, Sivas/ TURKEY

Abstract

The spectrum and spectral divisions of band matrices are very new and popular topics of studies. In this paper, our aims are to investigate boundedness of Jacobi matrix which is a band matrix has important role in physics and give subdivisions of the spectra, which are approximate point spectrum, defect spectrum and compression spectrum, for a special type Jacobi matrix. Moreover, we will find the fine division of spectrum which is given by Goldberg with the help of it.

Article info

History:

Received: 22.02.2021

Accepted: 10.08.2021

Keywords:

Jacobi matrices,
Spectrum,
Fine spectrum.

1. Introduction

The band matrices are an interesting topic for researchers since they have important applications in applied mathematics. In the summability theory and functional analysis, there are applications of band matrices. Also, they are used in linear algebra, computation in classical and fractional situations and approximation theory. The spectrum and spectral divisions of band matrices are very new and popular topics of studies.

In recent years, some authors have investigated the spectral decomposition of generalized difference matrices on various sequence spaces. In 2011, Amirov, Durna and Yıldırım [1] calculated the approximate point spectrum, the defect spectrum, and the compression spectrum of the operators using the relationship between the spectral decompositions of the operators. Many researchers have benefited from this study and found the fine division of the operator. In the studies conducted so far, the approximate point spectrum, the defect spectrum and the compression spectrum were calculated using the fine spectrum of the operator. Generally, in order to examine the fine spectrum of operator, we investigate injectivity and surjectivity of its adjoint. Because it is well-known that " T has a dense range if and only if T^* is 1-1" and " T has a bounded inverse if and only if T^* is onto". But we can not always find adjoint operator. Even if we find it, we can not investigate the character of the

series obtained while examining the injectivity and surjectivity of the adjoint operator. For example, it is not possible to talk about the adjoint of operator in general on ℓ_∞ , because ℓ_∞ does not have the Schauder basis in the usual sense. And so, we will first calculate the approximate point spectrum, the defect spectrum and the compression spectrum of operator using the relationship between spectral division of operator and spectral division of its adjoint. Moreover, we will find the fine division of spectrum which is given by Goldberg with the help of it.

Firstly, we will recall basic definitions and properties of operator which are used by us.

Definition 1.1 Let $T: D(T) \rightarrow X$ be a linear operator, defined on $D(T) \subset X$, where $D(T)$ denote the domain of T and X is an infinite-dimensional complex normed space. Let $T_\lambda := \lambda I - T$ for $T \in B(X)$ and $\lambda \in \mathbb{C}$ where I is the identity operator, then different definitions and notations of spectra are defined as follows [2-3]:

- (1) The spectrum: $\sigma(T, X) := \{\lambda \in \mathbb{C}: T_\lambda \text{ is not invertible}\},$
- (2) The resolvent set $\rho(T, X)$ is the complement of $\sigma(T, X)$ in $\mathbb{C},$
- (3) The point spectrum: $\sigma_p(T, X) := \{\lambda \in \mathbb{C}: T_\lambda \text{ is not injective}\},$
- (4) The continuous spectrum: $\sigma_c(T, X) := \{\lambda \in \mathbb{C}: T_\lambda \text{ is injective and } \overline{R(T_\lambda)} = X \text{ but } R(T_\lambda) \neq X\},$ where $R(T_\lambda)$ denote the domain of $T_\lambda,$

*Corresponding author. e-mail address: ndurna@cumhuriyet.edu.tr

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- (5) The residual spectrum: $\sigma_r(T, X) := \{\lambda \in \mathbb{C} : T_\lambda \text{ is injective but } \overline{R(T_\lambda)} \neq X\}$,
 (6) The defect spectrum: $\sigma_\delta(T, X) := \{\lambda \in \sigma(T, X) : R(T_\lambda) \neq X\}$,
 (7) The compression spectrum: $\sigma_{co}(T, X) := \{\lambda \in \mathbb{C} : \overline{R(T_\lambda)} \neq X\}$,
 (8) The approximate point spectrum: $\sigma_{ap}(T, X) := \{\lambda \in \mathbb{C} : \text{there exists a sequence } (x_n) \text{ in } X \text{ such that } \|x_n\| = 1 \text{ for all } n \in \mathbb{N} \text{ and } \lim_{n \rightarrow \infty} \|T_\lambda(x_n)\| = 0\}$.

In Banach spaces, Proposition 1.2 is frequently used for calculating the partition of the spectrum of the linear operator.

Proposition 1.2 [2] The spectra and subspectra of an operator $T \in B(X)$ and its adjoint $T^* \in B(X^*)$ are related by the following relations:

- (a) $\sigma(T^*, X^*) = \sigma(T, X)$,
 (b) $\sigma_c(T^*, X^*) \subseteq \sigma_{ap}(T, X)$,
 (c) $\sigma_{ap}(T^*, X^*) = \sigma_\delta(T, X)$,
 (d) $\sigma_\delta(T^*, X^*) = \sigma_{ap}(T, X)$,
 (e) $\sigma_p(T^*, X^*) = \sigma_{co}(T, X)$,
 (f) $\sigma_{co}(T^*, X^*) \supseteq \sigma_p(T, X)$,
 (g) $\sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_p(T^*, X^*) = \sigma_p(T, X) \cup \sigma_{ap}(T^*, X^*)$.

1.1. Goldberg's classification of spectrum

If X is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$:

- (I) $R(T) = X$, (II) $\overline{R(T)} = X$, but $R(T) \neq X$,
 (III) $\overline{R(T)} \neq X$

and three possibilities for T^{-1} :

- (1) T^{-1} exists and continuous,

(2) T^{-1} exists but discontinuous,

(3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: $I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3$. If an operator is in state III_2 for example, then $\overline{R(T)} \neq X$ and T^{-1} exist but is discontinuous (see [4]).

If λ is a complex number such that $T_\lambda \in I_1$ or $T_\lambda \in II_1$, then $\lambda \in \rho(T, X)$. All scalar values of λ not in $\rho(T, X)$ comprise the spectrum of T . The further classification of $\sigma(T, X)$ gives rise to the fine spectrum of T . That is, $\sigma(T, X)$ can be divided into the subsets $I_2\sigma(T, X) = \emptyset, I_3\sigma(T, X), II_2\sigma(T, X), II_3\sigma(T, X), III_1\sigma(T, X), III_2\sigma(T, X), III_3\sigma(T, X)$. For example, if T_λ is in a given state, III_2 (say), then we write $\lambda \in III_2\sigma(T, X)$.

Let us give a short survey concerning the spectrum and the fine spectrum and subdivision of the spectrum of the linear operators over certain sequence spaces.

First, the spectrum of the Cesàro operator of order one over the sequence space ℓ_2 has been examined by Brown, Halmos, and Shields [5] in 1965. In 1977, Cass and Rhoades [6], in 1978, Cardlidge [7] computed the spectrum of Weighted mean matrices.

Subdivisions of the spectrum for an operator on a sequence space were given by [8], [9] and [10] firstly.

Besides the above listed workers, the spectrum, fine spectrum and subdivision of the spectrum for various matrix operators have been investigated by many authors in the recent years, [11-25].

By the definitions given above, the following statements are obtained from the Table given by Durna and Yıldırım in [9]:

Table 1. Subdivisions of the spectrum of a linear operator

		1	2	3
		T_h^{-1} exists and is bounded	T_h^{-1} exists and is unbounded	T_h^{-1} does not exists
I	$R(T_h) = X$	$\lambda \in \rho(T, X)$ $\lambda \in \rho(T, X)$	–	$\lambda \in \sigma_p(T, X)$ $\lambda \in \sigma_{ap}(T, X)$
II	$\overline{R(T_h)} = X$	$\lambda \in \rho(T, X)$	$\lambda \in \sigma_c(T, X)$ $\lambda \in \sigma_{ap}(T, X)$ $\lambda \in \sigma_\delta(T, X)$	$\lambda \in \sigma_p(T, X)$ $\lambda \in \sigma_{ap}(T, X)$ $\lambda \in \sigma_\delta(T, X)$
III	$\overline{R(T_h)} \neq X$	$\lambda \in \sigma_r(T, X)$ $\lambda \in \sigma_\delta(T, X)$ $\lambda \in \sigma_{co}(T, X)$	$\lambda \in \sigma_r(T, X)$ $\lambda \in \sigma_{ap}(T, X)$ $\lambda \in \sigma_\delta(T, X)$ $\lambda \in \sigma_{co}(T, X)$	$\lambda \in \sigma_p(T, X)$ $\lambda \in \sigma_{ap}(T, X)$ $\lambda \in \sigma_\delta(T, X)$ $\lambda \in \sigma_{co}(T, X)$

In this paper, we computed subdivisions of the spectrum for constant Jacobi matrix.

2. Boundedness of Jacobi Matrix $J(s_n, r_n)$

A matrix of the form $J = (a_{ij})$ is called a Jacobi matrix, where $a_{ij} = 0$ unless $|j - i| < 1$. More specifically,

$$J(s_n, r_n) = \begin{pmatrix} s_0 & r_0 & 0 & 0 & \cdots \\ r_0 & s_1 & r_1 & 0 & \cdots \\ 0 & r_1 & s_2 & r_2 & \cdots \\ 0 & 0 & r_2 & s_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (1)$$

where all s_n, r_n are real. If we get some constant sequences such as $(s_n) = (s)$ and $(r_n) = (r)$, this $J(s_n, r_n) = J(s, r)$ matrix is called constant Jacobi matrix. The spectral results are clear when $r = 0$, so for the sequel we will have $r \neq 0$.

Lemma 2.1 [25] Let T be an operator with the associated matrix $A = (a_{nk})$. Then the followings hold:

i. $T \in B(c)$ if and only if

$$\|A\| := \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty, \quad (2)$$

$$a_k := \lim_{n \rightarrow \infty} a_{nk} \text{ exists for each } k, \quad (3)$$

$$a := \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} a_{nk} \text{ exists} \quad (4)$$

are valid.

ii. $T \in B(c_0)$ if and only if (2) and (3) with $a_k = 0$ for each k are valid.

iii. $T \in B(\ell_\infty)$ if and only if (2) is valid.

In these cases, the operator norm of T is

$$\|T\|_{(\ell_\infty: \ell_\infty)} = \|T\|_{(c:c)} = \|T\|_{(c_0:c_0)} = \|A\|. \quad (5)$$

iv. $T \in B(\ell_1)$ if and only if

$$\|A^t\| := \sup_k \sum_{n=1}^{\infty} |a_{nk}| < \infty \quad (6)$$

is valid.

In these cases, the operator norm of T is $\|T\|_{(\ell_1, \ell_1)} = \|A^t\|$.

Theorem 2.2 $\mu \in \{c_0, c, \ell_1, \ell_\infty\}$. $J(s, r) \in B(\mu)$ and $\|J(s, r)\|_{(\mu; \mu)} \leq 2|r| + |s|$.

Proof It is clear from Lemma 2.1.

Theorem 2.3 $J(s, r) \in B(\ell_p)$ ($1 < p < \infty$) and $\|J(s, r)\|_{(\ell_p; \ell_p)} \leq 2|r| + |s|$.

Proof Since

$$\begin{aligned} \|J(s, r)x\|_{\ell_p} &= \left(\sum_{n=1}^{\infty} |rx_{n-1} + sx_n + rx_{n+1}|^p \right)^{\frac{1}{p}} \\ &= \left(\sum_{n=1}^{\infty} |r(x_{n-1} + x_{n+1}) + sx_n|^p \right)^{\frac{1}{p}} \\ &\leq |r| \left(\sum_{n=1}^{\infty} |x_{n-1} + x_{n+1}|^p \right)^{\frac{1}{p}} + |s| \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} \\ &\leq 2|r|\|x\|_{\ell_p} + |s|\|x\|_{\ell_p} \leq (2|r| + |s|)\|x\|_{\ell_p}, \end{aligned}$$

where $x_0 = 0$, we have $J(s, r) \in B(\ell_p)$ and $\|J(s, r)\|_{(\ell_p; \ell_p)} \leq 2|r| + |s|$.

Theorem 2.4 $J(s, r) \in B(bv_p)$ ($1 < p < \infty$) and $\|J(s, r)\|_{(bv_p; bv_p)} \leq |s - r| + 3|r|$.

Proof We have

$$\begin{aligned} \|J(s, r)x\|_{bv_p}^p &= |rx_1 + sx_2 + rx_3 - sx_1 - rx_2|^p + |rx_2 + sx_3 + rx_4 - rx_1 - sx_2 - rx_3|^p + \dots \\ &= |(r-s)x_1 + (s-r)x_2 + rx_3|^p + |(r-s)x_2 + (s-r)x_3 + rx_4 - rx_1|^p + \dots \\ &\leq \sum_{n=0}^{\infty} |(r-s)x_{n+1} + (s-r)x_{n+2} + r(x_{n+3} - x_{n+2} + x_{n+2} - x_{n+1} + x_{n+1} - x_n)|^p \\ &= \left[\left(\sum_{n=0}^{\infty} |(s-r)(x_{n+2} - x_{n+1}) + r(x_{n+3} - x_{n+2} + x_{n+2} - x_{n+1} + x_{n+1} - x_n)|^p \right)^{1/p} \right]^p \\ &\leq \left[\left(\sum_{n=0}^{\infty} |s-r|^p |x_{n+2} - x_{n+1}|^p \right)^{1/p} \right. \\ &\quad \left. + \left(\sum_{n=0}^{\infty} |r|^p |x_{n+3} - x_{n+2} + x_{n+2} - x_{n+1} + x_{n+1} - x_n|^p \right)^{1/p} \right]^p \\ &\leq \left[\left(|s-r| \sum_{n=0}^{\infty} |x_{n+2} - x_{n+1}|^p \right)^{1/p} \right. \\ &\quad \left. + \left(|r| \sum_{n=0}^{\infty} (|x_{n+3} - x_{n+2}| + |x_{n+2} - x_{n+1}| + |x_{n+1} - x_n|)^p \right)^{1/p} \right]^p \end{aligned}$$

$$\begin{aligned}
&\leq \left[|s-r| \left(\sum_{n=0}^{\infty} |x_{n+2}-x_{n+1}|^p \right)^{1/p} \right. \\
&\quad \left. + |r| \left[\left(\sum_{n=0}^{\infty} |x_{n+3}-x_{n+2}|^p \right)^{1/p} + \left(\sum_{n=0}^{\infty} |x_{n+2}-x_{n+1}|^p \right)^{1/p} + \left(\sum_{n=0}^{\infty} |x_{n+1}-x_n|^p \right)^{1/p} \right] \right]^p \\
&\leq \left[|s-r| \|x\|_{bv_p} + |r| 3 \|x\|_{bv_p} \right]^p = [|s-r| + 3|r|]^p \|x\|_{bv_p}^p
\end{aligned}$$

where $x_0 = 0$. Then

$$\|J(s, r)x\|_{bv_p} \leq (|s-r| + 3|r|) \|x\|_{bv_p}.$$

Hence we get $J(s, r) \in B(bv_p)$ and $\|J(s, r)\|_{(bv_p: bv_p)} \leq |s-r| + 3|r|$.

Theorem 2.5 $J(s_n, r_n) \in B(\mu)$ and $\|J(s_n, r_n)\|_{(\mu: \mu)} \leq 2\|r\|_{\infty} + \|s\|_{\infty}$ where $\mu \in \{c_0, c, \ell_1, \ell_{\infty}\}$, $(s_n), (r_n) \in \mu$.

Proof It is clear from Lemma 2.1.

Theorem 2.6 $J(s_n, r_n) \in B(\ell_p)$ ($1 < p < \infty$) and $\|J(s_n, r_n)\|_{(\ell_p: \ell_p)} \leq 2\|r\|_p + \|s\|_p$ where $(s_n), (r_n) \in \ell_p$.

Proof Since

$$\begin{aligned}
\|J(s_n, r_n)x\|_p &= \left(\sum_{n=1}^{\infty} |r_{n-1}x_{n-1} + s_nx_n + r_nx_{n+1}|^p \right)^{\frac{1}{p}} \\
&\leq \left(\sum_{n=1}^{\infty} |r_{n-1}x_{n-1} + s_nx_n|^p \right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |r_nx_{n+1}|^p \right)^{\frac{1}{p}} \\
&\leq \left(\sum_{n=1}^{\infty} |r_{n-1}x_{n-1}|^p \right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |s_nx_n|^p \right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |r_nx_{n+1}|^p \right)^{\frac{1}{p}} \\
&\leq (2\|r\|_p + \|s\|_p) \|x\|_p,
\end{aligned}$$

we have $J(s_n, r_n) \in B(\ell_p)$ and $\|J(s_n, r_n)\|_{(\ell_p: \ell_p)} \leq 2\|r\|_p + \|s\|_p$.

3. Spectrum of Jacobi Matrix $J(s, r)$ with constant entries

In this section, we will give the spectral decomposition of Jacobi Matrix $J(s, r)$ with constant entries with the help of the spectrum and the fine spectrum, which were previously studied in [27] and [18].

3.1. Subdivision of the spectrum of $J(s, r)$ on c_0

Theorem 3.1 $\sigma_{ap}(J(s, r), c_0) = \sigma(J(s, r), c_0) = [s - 2r, s + 2r]$.

Proof From Table 1, we know

$$\sigma_{ap}(J(s, r), c_0) = \sigma(J(s, r), c_0) \setminus III_1 \sigma(J(s, r), c_0).$$

Since $\sigma_r(J(s, r), c_0) = \emptyset$ from [27, Theorem 3.3], we have $III_1\sigma(J(s, r), c_0) = \emptyset$ and we know $\sigma(J(s, r), c_0) = [s - 2r, s + 2r]$ from [27, Theorem 2.5]. Hence $\sigma_{ap}(J(s, r), c_0) = [s - 2r, s + 2r]$.

Theorem 3.2 $\sigma_\delta(J(s, r), c_0) = \sigma(J(s, r), c_0) = [s - 2r, s + 2r]$.

Proof We have

$$\sigma_\delta(J(s, r), c_0) = \sigma(J(s, r), c_0) \setminus I_3\sigma(J(s, r), c_0)$$

from Table 1. Since $\sigma_p(J(s, r), c_0) = \emptyset$ from [27, Theorem 2.1], we get $I_3\sigma(J(s, r), c_0) = \emptyset$ and we know $\sigma(J(s, r), c_0) = [s - 2r, s + 2r]$ from [27, Theorem 2.5]. Hence $\sigma_\delta(J(s, r), c_0) = [s - 2r, s + 2r]$.

Theorem 3.3 $\sigma_{co}(J(s, r), c_0) = \emptyset$.

Proof From Table 1, we get

$$\sigma_{co}(J(s, r), c_0) = III_1\sigma(J(s, r), c_0) \cup III_2\sigma(J(s, r), c_0) \cup III_3\sigma(J(s, r), c_0).$$

Since $\sigma_p(J(s, r), c_0) = \emptyset$ from [27, Theorem 2.1] and $\sigma_r(J(s, r), c_0) = \emptyset$ from [27, Theorem 3.3], we have

$$III_1\sigma(J(s, r), c_0) = III_2\sigma(J(s, r), c_0) = III_3\sigma(J(s, r), c_0) = \emptyset.$$

Therefore $\sigma_{co}(J(s, r), c_0) = \emptyset$.

3.2. The fine spectrum and subdivision of the spectrum of $J(s, r)$ on c

Theorem 3.4 $III_2\sigma(J(s, r), c) = \{s + 2r\}$.

Proof By [27, Theorem 3.5], $\sigma_r(J(s, r), c) = \{s + 2r\}$. So that $s + 2r \in III_1\sigma(J(s, r), c) \cup III_2\sigma(J(s, r), c)$. Now we investigate either $s + 2r \in III_1\sigma(J(s, r), c)$ or $s + 2r \in III_2\sigma(J(s, r), c)$. For this we must show that $((s + 2r)I - J(s, r))^{-1}$ is whether bounded or not.

$$\begin{aligned} (s + 2r)I - J(s, r) &= \begin{pmatrix} s + 2r & 0 & 0 & \cdots \\ 0 & s + 2r & 0 & \cdots \\ 0 & 0 & s + 2r & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} - \begin{pmatrix} s & r & 0 & 0 & \cdots \\ r & s & r & 0 & \cdots \\ 0 & r & s & r & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\ &= r \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots \\ -1 & 2 & -1 & 0 & \cdots \\ 0 & -1 & 2 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \end{aligned}$$

The inverse of above matrix is

$$\frac{1}{r} \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 2 & 2 & \cdots \\ 1 & 2 & 3 & 3 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which is unbounded matrix. Therefore we get $s + 2r \in III_2\sigma(J(s, r), c)$. Hence $III_2\sigma(J(s, r), c) = \{s + 2r\}$, from $III_2\sigma(J(s, r), c) \subset \sigma_r(J(s, r), c)$.

Corollary 3.5 $III_1\sigma(J(s, r), c) = \emptyset$.

Proof By [27, Theorem 3.5], $\sigma_r(J(s, r), c) = \{s + 2r\}$ and $III_1\sigma(J(s, r), c) = \emptyset$ from Theorem 3.4

Corollary 3.6 $\sigma_{ap}(J(s, r), c) = [s - 2r, s + 2r]$.

Proof From Table 1, we get

$$\sigma_{ap}(J(s, r), c) = \sigma(J(s, r), c) \setminus III_1 \sigma(J(s, r), c).$$

Since $III_1 \sigma(J(s, r), c) = \emptyset$ from Corollary 3.5 and $\sigma(J(s, r), c) = [s - 2r, s + 2r]$ from [27, Theorem 2.5], we have $\sigma_{ap}(J(s, r), c) = [s - 2r, s + 2r]$.

Corollary 3.7 $\sigma_{\delta}(J(s, r), c) = [s - 2r, s + 2r]$.

Proof We know

$$\sigma_{\delta}(J(s, r), c) = \sigma(J(s, r), c) \setminus I_3 \sigma(J(s, r), c)$$

from Table 1. Since $\sigma_p(J(s, r), c) = \emptyset$ from [27, Theorem 2.1] and $\sigma(J(s, r), c) = [s - 2r, s + 2r]$ from [27, Theorem 2.5], we get $\sigma_{\delta}(J(s, r), c) = [s - 2r, s + 2r]$.

Corollary 3.8 $\sigma_{co}(J(s, r), c) = \{s + 2r\}$.

Proof By Table 1, we get

$$\sigma_{co}(J(s, r), c) = III_1 \sigma(J(s, r), c) \cup III_2 \sigma(J(s, r), c) \cup III_3 \sigma(J(s, r), c).$$

Using $\sigma_p(J(s, r), c) = \emptyset$ from [27, Theorem 2.1], we get $\sigma_{co}(J(s, r), c) = \{s + 2r\}$ from Theorem 3.4 and Corollary 3.5.

3.3. The fine spectrum and subdivision of the spectrum of $J(s, r)$ on ℓ_1

Theorem 3.9 $III_1 \sigma(J(s, r), \ell_1) = \emptyset$.

Proof We know

$$\sigma_{ap}(J^*(s, r), c^* \equiv \ell_1) = \sigma(J^*(s, r), \ell_1) \setminus III_1 \sigma(J^*(s, r), \ell_1)$$

by Table 1 and we have

$$\sigma_{ap}(J^*(s, r) = J(s, r), \ell_1) = \sigma_{\delta}(J(s, r), c_0)$$

by the Proposition 1.2. We get

$$\sigma_{\delta}(J(s, r), c_0) = \sigma(J(s, r), c_0) \setminus I_3 \sigma(J(s, r), c_0)$$

by Table 1. Since $\sigma(J(s, r), c_0) = [s - 2r, s + 2r]$ from [27, Theorem 2.5], and $I_3 \sigma(J(s, r), c_0) = \emptyset$, we have $\sigma_{\delta}(J(s, r), c_0) = [s - 2r, s + 2r]$. Therefore $\sigma_{ap}(J^*(s, r) = J(s, r), \ell_1) = [s - 2r, s + 2r]$ and since $\sigma(J^*(s, r), \ell_1) = \sigma(J(s, r), \ell_1) = [s - 2r, s + 2r]$ from Proposition 1.2, we get $III_1 \sigma(J(s, r), \ell_1) = \emptyset$.

Corollary 3.10 $III_2 \sigma(J(s, r), \ell_1) = (s - 2r, s + 2r)$.

Proof We know $\sigma_r(J(s, r), \ell_1) = (s - 2r, s + 2r)$ by [27, Theorem 3.4] and we have $III_2 \sigma(J(s, r), \ell_1) = (s - 2r, s + 2r)$ by Theorem 3.9.

Corollary 3.11 $\sigma_{ap}(J(s, r), \ell_1) = [s - 2r, s + 2r]$.

Proof From Table 1, we know

$$\sigma_{ap}(J(s, r), \ell_1) = \sigma(J(s, r), \ell_1) \setminus III_1 \sigma(J(s, r), \ell_1).$$

Since $\sigma(J(s, r), \ell_1) = [s - 2r, s + 2r]$ from [27, Theorem 2.5] and $III_1 \sigma(J(s, r), \ell_1) = \emptyset$ from Theorem 3.9, we have $\sigma_{ap}(J(s, r), \ell_1) = [s - 2r, s + 2r]$.

Corollary 3.12 $\sigma_\delta(J(s, r), \ell_1) = [s - 2r, s + 2r]$.

Proof Since $\sigma_p(J(s, r), \ell_1) = \emptyset$ from [27, Theorem 2.1], we have $I_3 \sigma(J(s, r), \ell_1) = \emptyset$. And since

$$\sigma_\delta(J(s, r), \ell_1) = \sigma(J(s, r), \ell_1) \setminus I_3 \sigma(J(s, r), \ell_1)$$

by Table 1, we have $\sigma_\delta(J(s, r), \ell_1) = [s - 2r, s + 2r]$.

Corollary 3.13 $\sigma_{co}(J(s, r), \ell_1) = (s - 2r, s + 2r)$.

Proof From Table 1, we have

$$\sigma_{co}(J(s, r), \ell_1) = III_1 \sigma(J(s, r), \ell_1) \cup III_2 \sigma(J(s, r), \ell_1) \cup III_3 \sigma(J(s, r), \ell_1).$$

Using $\sigma_r(J(s, r), \ell_1) = (s - 2r, s + 2r)$ from [27, Theorem 3.4] and $\sigma_p(J(s, r), \ell_1) = \emptyset$ from [27, Theorem 2.1], we have $\sigma_{co}(J(s, r), \ell_1) = (s - 2r, s + 2r)$ from Theorem 3.9 and Corollary 3.10.

3.4. Subdivision of the spectrum of $J(s, r)$ on ℓ_p

Theorem 3.14 $\sigma_{ap}(J(s, r), \ell_p) = \sigma(J(s, r), \ell_p) = [s - 2r, s + 2r]$.

Proof We know

$$\sigma_{ap}(J(s, r), \ell_p) = \sigma(J(s, r), \ell_p) \setminus III_1 \sigma(J(s, r), \ell_p)$$

by the Table 1. Since $\sigma_r(J(s, r), \ell_p) = \emptyset$ from [18, Corollary 3.5], we find $III_1 \sigma(J(s, r), \ell_p) = \emptyset$ and we have $\sigma(J(s, r), \ell_p) = [s - 2r, s + 2r]$ from [18, Theorem 3.2]. Therefore $\sigma_{ap}(J(s, r), \ell_p) = [s - 2r, s + 2r]$.

Theorem 3.15 $\sigma_\delta(J(s, r), \ell_p) = \sigma(J(s, r), \ell_p) = [s - 2r, s + 2r]$.

Proof From Table 1, we have

$$\sigma_\delta(J(s, r), \ell_p) = \sigma(J(s, r), \ell_p) \setminus I_3 \sigma(J(s, r), \ell_p).$$

We have $I_3 \sigma(J(s, r), \ell_p) = \emptyset$ and $\sigma(J(s, r), \ell_p) = [s - 2r, s + 2r]$ from [18, Theorem 3.2] since $\sigma_p(J(s, r), \ell_p) = \emptyset$ from [18, Theorem 3.3]. Hence $\sigma_\delta(J(s, r), \ell_p) = [s - 2r, s + 2r]$.

Theorem 3.16 $\sigma_{co}(J(s, r), \ell_p) = \emptyset$.

Proof We know

$$\sigma_{co}(J(s, r), \ell_p) = III_1 \sigma(J(s, r), \ell_p) \cup III_2 \sigma(J(s, r), \ell_p) \cup III_3 \sigma(J(s, r), \ell_p)$$

by the Table 1. Since $\sigma_p(J(s, r), \ell_p) = \emptyset$ from [18, Theorem 3.3] and $\sigma_r(J(s, r), \ell_p) = \emptyset$ from [18, Corollary 3.5], we get $III_1 \sigma(J(s, r), \ell_p) = III_2 \sigma(J(s, r), \ell_p) = III_3 \sigma(J(s, r), \ell_p) = \emptyset$. Hence $\sigma_{co}(J(s, r), \ell_p) = \emptyset$.

3.5. The fine spectrum and subdivision of the spectrum of $J(s, r)$ on ℓ_∞

Theorem 3.17 $I_3\sigma(J(s, r), \ell_\infty) = \emptyset$.

Proof From Table 1, we have

$$\sigma_\delta(J(s, r), \ell_\infty) = \sigma(J(s, r), \ell_\infty) \setminus I_3\sigma(J(s, r), \ell_\infty).$$

We get $\sigma_\delta(J(s, r), \ell_\infty) = \sigma_{ap}(J^*(s, r), \ell_1)$ from Proposition 1.2. Using Corollary 3.11, $\sigma_{ap}(J(s, r), \ell_1) = [s - 2r, s + 2r]$ and $\sigma(J(s, r), \ell_\infty) = [s - 2r, s + 2r]$ from [27, Theorem 2.5], we have $I_3\sigma(J(s, r), \ell_\infty) = \emptyset$.

Theorem 3.18 $III_1\sigma(J(s, r), \ell_\infty) = \emptyset$.

Proof We know

$$\sigma_{ap}(J(s, r), \ell_\infty) = \sigma(J(s, r), \ell_\infty) \setminus III_1\sigma(J(s, r), \ell_\infty)$$

by Table 1. $\sigma_{ap}(J^*(s, r), \ell_\infty) = \sigma_\delta(J(s, r), \ell_1)$ from Proposition 1.2. Since $\sigma_\delta(J(s, r), \ell_1) = [s - 2r, s + 2r]$ from Corollary 3.12 and $\sigma(J(s, r), \ell_\infty) = [s - 2r, s + 2r]$ from [27, Theorem 2.5], we have $III_1\sigma(J(s, r), \ell_\infty) = \emptyset$.

Corollary 3.19 $\sigma_{ap}(J(s, r), \ell_\infty) = [s - 2r, s + 2r]$.

Proof From Table 1, we get

$$\sigma_{ap}(J(s, r), \ell_\infty) = \sigma(J(s, r), \ell_\infty) \setminus III_1\sigma(J(s, r), \ell_\infty).$$

We have $\sigma_{ap}(J(s, r), \ell_\infty) = [s - 2r, s + 2r]$ since $\sigma(J(s, r), \ell_\infty) = [s - 2r, s + 2r]$ from [27 Theorem 2.5] and $III_1\sigma(J(s, r), \ell_\infty) = \emptyset$ from Theorem 3.18.

Corollary 3.20 $\sigma_\delta(J(s, r), \ell_\infty) = [s - 2r, s + 2r]$.

Proof We know

$$\sigma_\delta(J(s, r), \ell_\infty) = \sigma(J(s, r), \ell_\infty) \setminus I_3\sigma(J(s, r), \ell_\infty)$$

by the Table 1. Since $\sigma(J(s, r), \ell_\infty) = [s - 2r, s + 2r]$ from [27, Theorem 2.5] and $I_3\sigma(J(s, r), \ell_\infty) = \emptyset$ from Theorem 3.17, we have $\sigma_\delta(J(s, r), \ell_\infty) = [s - 2r, s + 2r]$.

3.6. Subdivision of the spectrum of $J(s, r)$ on bv_p

Theorem 3.21 $\sigma_{ap}(J(s, r), bv_p) = \sigma(J(s, r), bv_p) = [s - 2r, s + 2r]$.

Proof From Table 1, we get

$$\sigma_{ap}(J(s, r), bv_p) = \sigma(J(s, r), bv_p) \setminus III_1\sigma(J(s, r), bv_p).$$

Since $\sigma_r(J(s, r), bv_p) = \emptyset$ from [18, Theorem 4.3 (iii)], we have $III_1\sigma(J(s, r), bv_p) = \emptyset$ and $\sigma(J(s, r), bv_p) = [s - 2r, s + 2r]$ from [18, Theorem 4.2]. Therefore $\sigma_{ap}(J(s, r), bv_p) = [s - 2r, s + 2r]$.

Theorem 3.22 $\sigma_\delta(J(s, r), bv_p) = \sigma(J(s, r), bv_p) = [s - 2r, s + 2r]$.

Proof We know

$$\sigma_{\delta}(J(s, r), bv_p) = \sigma(J(s, r), bv_p) \setminus I_3 \sigma(J(s, r), bv_p)$$

by the Table 1. Since $\sigma_p(J(s, r), bv_p) = \emptyset$ from [18, Theorem 4.3 (i)], we have $I_3 \sigma(J(s, r), bv_p) = \emptyset$ and we have $\sigma(J(s, r), bv_p) = [s - 2r, s + 2r]$ from [18, Theorem 4.2]. Hence $\sigma_{\delta}(J(s, r), bv_p) = [s - 2r, s + 2r]$.

Theorem 3.16 $\sigma_{co}(J(s, r), bv_p) = \emptyset$.

Proof From Table 1, we get

$$\sigma_{co}(J(s, r), bv_p) = III_1 \sigma(J(s, r), bv_p) \cup III_2 \sigma(J(s, r), bv_p) \cup III_3 \sigma(J(s, r), bv_p)$$

Since $\sigma_p(J(s, r), bv_p) = \emptyset$ from [18, Theorem 4.3 (i)] and $\sigma_r(J(s, r), bv_p) = \emptyset$ from [18, Theorem 4.3 (iii)], we have $III_1 \sigma(J(s, r), bv_p) = III_2 \sigma(J(s, r), bv_p) = III_3 \sigma(J(s, r), bv_p) = \emptyset$. Hence $\sigma_{co}(J(s, r), bv_p) = \emptyset$.

Acknowledgment

This work is supported by the Scientific Research Project Fund of Sivas Cumhuriyet University under the project number F-449.

Conflicts of interest

The authors state that did not have a conflict of interests.

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