

The space of initial data for the Robin boundary-value problem for parabolic differential-difference equation

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Abstract. In this paper, the third boundary-value problem for parabolic differential-difference equation is considered in Lipschitz spatial domain Q . It is proved that the space of initial data (i.e., the space of initial functions for which the strong solution exists) coincides with Sobolev space $H^1(Q)$.

Key words. Differential-difference equations, the space of initial data, Kato's conjecture.

Parabolik diferensiyel-fark denklemleri için Robin sınır değer probleminin başlangıç değer uzayı

Özet. Bu makalede Q Lipschitz uzaysal tanım kümesinde parabolik fark denklemi için üçüncü sınır değer problemi incelenmiştir. Başlangıç veri uzayının (güçlü çözümlerin var olduğu başlangıç fonksiyonları uzayı) $H^1(Q)$ Sobolev uzayı ile çakıştığı kanıtlanmıştır.

Anahtar kelimeler. Diferensiyel-fark denklemleri, başlangıç değer uzayı, Kato varsayımı.

1 Introduction

The space of initial data, i.e., the space of initial functions for which the strong solution exists, for the second boundary-value problem for parabolic differential-difference equation in case of

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smooth spatial domain Q was described in terms of the Sobolev spaces in [7]. For the history of the problem and its connection with the Kato square root problem see [7] and literature therein.

In this paper we spread this result to the Robin boundary conditions and we use the method developed in [1] to cover the case of Lipschitz spatial domain Q .

2 Statement of the problem

Let $Q \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with Lipschitz boundary ∂Q , i.e., locally it is a graph of function $x_n = \phi(x')$, satisfying the Lipschitz condition $|\phi(x') - \phi(y')| \leq K|x' - y'|$ with $K > 0$.

Introduce bounded difference operators R_{ij} , $R_i: L_2(\mathbb{R}^n) \rightarrow L_2(\mathbb{R}^n)$ by the formulas

$$\begin{aligned} (R_{ij}u)(x) &= \sum_{h \in M} a_{ijh}u(x+h) \quad (i, j = 1, 2, \dots, n), \\ (R_iu)(x) &= \sum_{h \in M} a_{ih}u(x+h) \quad (i = 0, 1, 2, \dots, n). \end{aligned} \quad (2.1)$$

Here, a_{ijh} , a_{ih} are complex numbers, the set M consists of a finite number of vectors $h \in \mathbb{R}^n$ with integer coordinates.

We introduce the following linear operators: $I_Q: L_2(Q) \rightarrow L_2(\mathbb{R}^n)$ is the operator of extension of a function by zero outside Q ; $P_Q: L_2(\mathbb{R}^n) \rightarrow L_2(Q)$ is the projection operator of a function onto Q ; and the operators R_{ijQ} , $R_{iQ}: L_2(Q) \rightarrow L_2(Q)$ defined by the formulas $R_{ijQ} = P_Q R_{ij} I_Q$, $R_{iQ} = P_Q R_i I_Q$.

We consider the following differential-difference equation

$$u_t - \sum_{i,j=1}^n (R_{ijQ}u_{x_j})_{x_i} + \sum_{i=1}^n R_{iQ}u_{x_i} + R_{0Q}u = f(x, t) \quad ((x, t) \in Q_T) \quad (2.2)$$

with the boundary condition

$$\sum_{i,j=1}^n R_{ijQ}u_{x_j} \cos(\nu, x_i) + \sigma(x)u = 0 \quad ((x, t) \in \Gamma_T), \quad (2.3)$$

and the initial condition

$$u|_{t=0} = \varphi(x) \quad (x \in Q), \quad (2.4)$$

where $Q_T = Q \times (0, T)$, $0 < T < \infty$, $\Gamma_T = \partial Q \times (0, T)$, ν is the external unit normal to Γ_T (it exists at almost every point of Γ_T), $\sigma \in L_2(\partial Q)$ and $\sigma \geq 0$ on ∂Q , $f \in L_2(Q_T)$, and $\varphi \in L_2(Q)$.

Let $H^k(Q)$ be the Sobolev space of complex-valued functions from $L_2(Q)$ having all generalized derivatives up to the k -order from $L_2(Q)$.

Introduce the sesquilinear form $a[v, w]$ in $L_2(Q)$ with domain $H^1(Q)$ by the formula

$$a[v, w] = \sum_{i,j=1}^n (R_{ijQ} v_{x_j}, w_{x_i})_{L_2(Q)} + \sum_{i=1}^n (R_{iQ} v_{x_i}, w)_{L_2(Q)} + (R_{0Q} v, w)_{L_2(Q)} + (\sigma v, w)_{L_2(\partial Q)}. \quad (2.5)$$

The difference operators $R_{ijQ}, R_{iQ}, R_{0Q}: L_2(Q) \rightarrow L_2(Q)$ are bounded. Therefore, it follows that there exists a constant $c_0 > 0$ such that

$$|a[v, w]| \leq c_0 \|v\|_{H^1(Q)} \|w\|_{H^1(Q)} \quad (v, w \in H^1(Q)). \quad (2.6)$$

Since the sesquilinear form $a[v, w]$ is continuous with respect to w in $H^1(Q)$, there exists a linear bounded operator $A: H^1(Q) \rightarrow [H^1(Q)]'$, such that

$$\langle Av, \bar{w} \rangle = a[v, w] \quad (v, w \in H^1(Q)), \quad (2.7)$$

where $[H^1(Q)]'$ is a dual space to $H^1(Q)$.

Definition 2.1 *The form $a[v, w]$ is said to be coercive if there exist numbers $c_1 > 0$ and $c_2 \geq 0$ such that*

$$\operatorname{Re} a[v, v] \geq c_1 \|v\|_{H^1(Q)}^2 - c_2 \|v\|_{L_2(Q)}^2 \quad (v \in H^1(Q)). \quad (2.8)$$

For the necessary and sufficient conditions of coercivity in algebraic form see [8, Lemmas 2.2, 2.3] (they are truth also in case ∂Q is Lipschitz and $\sigma \in L_2(Q)$). Further we shall assume that the form $a[v, w]$ is coercive.

3 Strong solutions and the spaces of initial data

1. The sesquilinear form $a[v, w]$ is continuous with respect to w in $L_2(Q)$, therefore it defines an operator $\mathcal{A}: D(\mathcal{A}) \subset L_2(Q) \rightarrow L_2(Q)$ such that

$$\mathcal{A}v = Av \quad (v \in D(\mathcal{A})). \quad (3.1)$$

Here $\|v\|_{D(\mathcal{A})} = \|\mathcal{A}v\|_{L_2(Q)} + \|v\|_{L_2(Q)}$. Since the operator $\mathcal{A}$ is closed, the space $D(\mathcal{A})$ is a Hilbert space, moreover $D(\mathcal{A})$ is dense in $H^1(Q)$. It is well known that operator $-\mathcal{A}$ is a generator of an analytic contractive semigroup (sf. [4, Chapter IX, Section 1, Theorem 1.24]).

Introduce the Hilbert space

$$\mathcal{W} = \{w \in L_2(0, T; D(\mathcal{A})) : w_t \in L_2(Q_T)\}$$

with the norm

$$\|u\|_{\mathcal{W}}^2 = \int_0^T \|\mathcal{A}u\|_{L_2(Q)}^2 dt + \int_0^T \|u\|_{L_2(Q)}^2 dt + \int_0^T \|u_t\|_{L_2(Q)}^2 dt.$$

Here, the derivatives are considered in the sense of distributions on Q_T .

Define the bounded operator $L: \mathcal{W} \rightarrow L_2(0, T; L_2(Q))$ by the formula $Lv(\cdot, t) = \mathcal{A}v(\cdot, t)$ for almost all $t \in (0, T)$.

Definition 3.1 *A function $u \in \mathcal{W}$ is called a strong solution of Problem (2.2)–(2.4) if it satisfies the equation*

$$\frac{du}{dt} + Lu = f \quad \text{for almost all } t \in (0, T) \quad (3.2)$$

and the initial condition

$$u|_{t=0} = \varphi. \quad (3.3)$$

We can assume that $c_2 = 0$ in inequality (2.8). In opposite case, we set $u = z e^{c_2 t}$. Then, Problem (3.2)–(3.3) will be equivalent to the problem $\frac{dz}{dt} + (L + c_2 I)z = e^{-c_2 t} f$, $z|_{t=0} = \varphi$.

Theorem 3.2 *Let the form $a[v, w]$ be coercive and $c_2 = 0$. Then, Problem (2.2)–(2.4) for any $f \in L_2(Q_T)$ and $\varphi \in [L_2(Q), D(\mathcal{A})]_{1/2}$ has a unique strong solution given by the formula*

$$u(x, t) = T_t \varphi(x) + \int_0^t T_{t-s} f(x, s) ds, \quad (3.4)$$

where $\{T_t\}$ ($t \geq 0$) is the analytic semigroup generated by the operator $-\mathcal{A}$.

For the proof see, e.g., the proof of [8, Theorem 4.2].

2. Theorem 3.2 is connected with the problem on the space of initial data. In [2], it was proved that condition $\varphi \in [L_2(Q), D(\mathcal{A})]_{1/2}$ is necessary and sufficient for existence of strong solutions of Problem (2.2)–(2.4). Thus, there arises the problem of description of the interpolational space $[L_2(Q), D(\mathcal{A})]_{1/2}$.

Theorem 3.3 *Let the form $a[v, w]$ be coercive with $c_2 = 0$. Then,*

$$[L_2(Q), D(\mathcal{A})]_{1/2} = H^1(Q). \quad (3.5)$$

Proof. By [5, Theorem 3.1], $[L_2(Q), D(\mathcal{A})]_{1/2} = D(\mathcal{A}^{1/2})$. Therefore, we can use [1, Theorem 3.2] to prove equality (3.5).

It is enough to show that operator $A: H^{1+s}(Q) \rightarrow \tilde{H}^{-1+s}(Q)$ is bounded, when $|s| < 1/2$. But for convenience of the reader we give the complete proof. For $s \geq 0$ we denote by $H^s(Q)$ the Sobolev—Slobodetsky space $W_2^s(Q)$, and for $s < 0$ we denote by $\tilde{H}^s(Q)$ the dual space.

I. Consider the form $a[v, w]$ on $H^{1+s}(Q) \times H^{1-s}(Q)$:

$$a[v, w] = \sum_{i,j=1}^n \langle R_{ij,Q} v_{x_j}, \bar{w}_{x_i} \rangle + \sum_{i=1}^n \langle R_{i,Q} v_{x_i}, \bar{w} \rangle + (R_{0,Q} v, w)_{L_2(Q)} + (\sigma v, w)_{L_2(\partial Q)}.$$

If $v \in H^{1+s}(Q)$ and $w \in H^{1-s}(Q)$, then $v_{x_j} \in H^s(Q)$ and $w_{x_i} \in H^{-s}(Q)$. Recall that for $-1/2 < s < 0$ H^s coincides with $\tilde{H}^s$. The operators $R_{ij,Q}$, $R_{i,Q}$ are bounded in $H^s(Q)$, when $|s| < 1/2$, because the extension of a function by zero outside Q is a bounded operator from $H^s(Q)$ to $H^s(\mathbb{R}^n)$ (sf. [6, Theorem 3.33] for $s \geq 0$. For $s < 0$ it follows from equality $(\chi_Q \varphi, \psi)_{L_2(\mathbb{R}^n)} = (\varphi, \chi_Q \psi)_{L_2(\mathbb{R}^n)}$, where $\chi_Q(x) = 1$ if $x \in \bar{Q}$ and $\chi_Q(x) = 0$ otherwise and $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$). Thereby, for a fixed $|s| < 1/2$ there exists a constant $C_s > 0$ such that

$$|a[v, w]| \leq C_s \|v\|_{H^{1+s}(Q)} \|w\|_{H^{1-s}(Q)}.$$

Then the operator A is defined by the form:

$$\langle Av, \bar{w} \rangle = a[v, w], \quad v \in H^{1+s}(Q), w \in H^{1-s}(Q),$$

where $f = Av$ is from $\tilde{H}^{-1+s}(Q)$.

II. The form $\overline{a[w, v]}$ defines operators A^* and $\mathcal{A}^*$. From [10, Theorem 2.8], it follows that there exists $0 < \varepsilon < 1/2$ such that operators A and A^* on $H^{1+s}(Q)$ have bounded inverse, when $|s| < \varepsilon$. Let $0 < s < \varepsilon$, then by [3, Theorem 1.1] and reiteration theorem (sf. [9])

$$D(\mathcal{A}^{(1-s)/2}) = D(\mathcal{A}^{*(1-s)/2}) = D(S^{(1-s)/2}) = [L_2(Q), D(S^{1/2})]_{1-s} = H^{1-s}(Q). \quad (3.6)$$

Here operators S and $\mathcal{S}$ are defined by the form $\frac{1}{2} (a[v, w] + \overline{a[w, v]})$. Therefore, operator $\mathcal{A}^{*(1-s)/2}$ is bounded from $L_2(Q)$ to $H^{1-s}(Q)$ and there exists $C_1 > 0$ such that

$$\|S^{1/2} \mathcal{A}^{*(1-s)/2} w\|_{H^{-s}(Q)} \leq C_1 \|w\|_{L_2(Q)} \quad (w \in L_2(Q)). \quad (3.7)$$

III. Set

$$B = S^{-1/2} A S^{-1/2}. \quad (3.8)$$

Operator B is a bounded operator in $H^{\pm s}(Q)$. Then

$$A = S^{1/2} B S^{1/2} \quad (3.9)$$

and there exists $C_2 > 0$ such that

$$\|BS^{1/2}v\|_{H^s(Q)} = \|S^{-1/2}Av\|_{H^s(Q)} \leq C_2\|v\|_{H^{1+s}(Q)} \quad (v \in H^{1+s}(Q)). \quad (3.10)$$

IV. Operator A is one-to-one and bounded from $D(\mathcal{A})$ and $H^{1+s}(Q)$ to $L_2(Q)$ and $\tilde{H}^{-1+s}(Q)$ respectively. Since $L_2(Q) \subset \tilde{H}^{-1+s}(Q)$ densely and continuously, then $D(\mathcal{A}) \subset H^{1+s}(Q)$ densely and continuously.

For functions $v \in D(\mathcal{A})$ and $w \in D(\mathcal{A}^*)$ we have

$$(v, \mathcal{A}^{*(1+s)/2}w)_{L_2(Q)} = (v, \mathcal{A}^* \mathcal{A}^{*(1-s)/2}w)_{L_2(Q)} = (\mathcal{A}v, \mathcal{A}^{*(1-s)/2}w)_{L_2(Q)} \quad (3.11)$$

or

$$(\mathcal{A}^{(1+s)/2}v, w)_{L_2(Q)} = (Av, \mathcal{A}^{*(1-s)/2}w)_{L_2(Q)} = (BS^{1/2}v, S^{1/2}\mathcal{A}^{*(1-s)/2}w)_{L_2(Q)}. \quad (3.12)$$

By (3.7), (3.10), and Schwarz inequality, it follows that

$$\begin{aligned} \left| (\mathcal{A}^{(1+s)/2}v, w)_{L_2(Q)} \right| &\leq C_3 \|BS^{1/2}v\|_{H^s(Q)} \|S^{1/2}\mathcal{A}^{*(1-s)/2}w\|_{H^{-s}(Q)} \\ &\leq C_4 \|v\|_{H^{1+s}(Q)} \|w\|_{L_2(Q)}. \end{aligned} \quad (3.13)$$

Since $D(\mathcal{A}^*)$ is dense in $L_2(Q)$,

$$\|\mathcal{A}^{(1+s)/2}v\|_{L_2(Q)} \leq C_4 \|v\|_{H^{1+s}(Q)}. \quad (3.14)$$

The last inequality is true for $v \in H^{1+s}(Q)$ because $D(\mathcal{A}_2)$ is dense in $H^{1+s}(Q)$. Therefore, $H^{1+s}(Q) \subset D(\mathcal{A}^{(1+s)/2})$.

V. By reiteration theorem (sf. [9])

$$H^1(Q) = [L_2(Q), H^{1+s}(Q)]_{1/(1+s)} \subset [L_2(Q), D(\mathcal{A}^{(1+s)/2})]_{1/(1+s)} = D(\mathcal{A}^{1/2}). \quad (3.15)$$

The same is true for $\mathcal{A}^*$. Then equality (3.5) follows from [5, Corollary 5.1]. ■

Corollary 3.4 *If the form $a[v, w]$ is coercive, then $D(\mathcal{A}^{1/2}) = D(\mathcal{A}^{*1/2}) = H^1(Q)$.*

Remark 3.5 *The boundary condition (2.3) satisfies under additional smoothness conditions (see [8, Corollary 5.1]). In general case it must be understood in the sense of the Green inequality.*

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