

Weak and strong solvability in L_p spaces of the second boundary-value problem for a parabolic differential-difference equation

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Abstract. In this paper, the weak and strong solvability of the second boundary-value problem for a parabolic differential-difference equation is considered in spaces like $L_p(0, T; X)$. The spaces of initial data are described in terms of Besov spaces.

Key words. Differential-difference equations, the space of initial data, maximal regularity.

L_p uzayında parabolik diferensiyel-fark denklemi için ikinci sınır-değer probleminin zayıf ve güçlü çözülebilirliği

Özet. Bu makalede parabolik diferensiyel-fark denklemi için ikinci sınır- değer probleminin zayıf ve güçlü çözülebilirliği $L_p(0, T; X)$ gibi uzaylarda ele alınmıştır. Başlangıç değer uzayları Besov uzayları kullanılarak ifade edilmiştir.

Anahtar kelimeler. Diferensiyel-fark denklemleri, başlangıç değer uzayı, maksimal regülerlik.

1 Introduction

In this paper, we consider the following second boundary-value problem for a parabolic differential-difference equation in a bounded domain $Q \subset \mathbb{R}^n$ with Lipschitz boundary ∂Q

$$u_t - \sum_{i,j=1}^n (R_{ij}Q u_{x_j})_{x_i} + \sum_{i=1}^n R_{iQ} u_{x_i} + R_{0Q} u = f(x, t) \quad ((x, t) \in Q_T) \quad (1.1)$$

with the boundary condition

$$\sum_{i,j=1}^n R_{ij}Q u_{x_j} \cos(\nu, x_i) = 0 \quad ((x, t) \in \Gamma_T), \quad (1.2)$$

and the initial condition

$$u|_{t=0} = \varphi(x) \quad (x \in Q), \quad (1.3)$$

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where $Q_T = Q \times (0, T)$, $0 < T < \infty$, $\Gamma_T = \partial Q \times (0, T)$, ν is the external unit normal to Γ_T (it exists at almost every point of Γ_T), $f \in L_2(Q_T)$, and $\varphi \in L_2(Q)$. The operators $R_{ij,Q}$, R_{iQ} , and R_{0Q} will be described later, it is only important that they are multipliers in spaces $H^s(Q)$ for small $|s|$.

Note that the second mixed problem for parabolic functional differential equations arises in the theory of nonlinear optical systems (see, e.g., references in [8]).

We study the weak solvability of problem (1.1)–(1.3) in $L_p(0, T; \tilde{H}^{-1}(Q))$ and strong solvability in $L_p(0, T; L_2(Q))$. The main idea is to associate an operator A in $\tilde{H}^{-1}(Q)$ (as well as its part in $L_2(Q)$ denoted by $\mathcal{A}$) to the elliptic part of the problem and to reach the abstract Cauchy problem

$$u' + Au = f, \quad u(0) = \varphi.$$

In order to apply well-known results [2, Ch. 1, Sec. 3], we have to prove that negative operators A and $\mathcal{A}$ generate analytic semigroups and describe the interpolation spaces $\left(\tilde{H}^{-1}(Q), H^1(Q)\right)_{1-\frac{1}{p}, p}$ and $(L_2(Q), D(\mathcal{A}))_{1-\frac{1}{p}, p}$. The first interpolation space can be described in terms of Besov spaces for $0 < p < \infty$, when the problem of description of the second space is that in general we do not know the space $D(\mathcal{A})$. But we show that it can be described for $1 < p \leq 2$, using the recently proved equality $D(\mathcal{A}^{1/2}) = H^1(Q)$ (see [9]).

For the history of the problem and its connection with the Kato square root problem see [8] and literature therein.

2 Weak solvability

Let us introduce bounded difference operators $R_{ij}, R_i: L_2(\mathbb{R}^n) \rightarrow L_2(\mathbb{R}^n)$ by the formulas

$$\begin{aligned} (R_{ij}u)(x) &= \sum_{h \in M} a_{ijh} u(x+h) \quad (i, j = 1, 2, \dots, n), \\ (R_i u)(x) &= \sum_{h \in M} a_{ih} u(x+h) \quad (i = 0, 1, 2, \dots, n). \end{aligned} \tag{2.1}$$

Here, a_{ijh}, a_{ih} are complex numbers, the set M consists of a finite number of vectors $h \in \mathbb{R}^n$ with integer coordinates.

We introduce the following linear operators: $I_Q: L_2(Q) \rightarrow L_2(\mathbb{R}^n)$ is the operator of extension of a function by zero outside Q ; $P_Q: L_2(\mathbb{R}^n) \rightarrow L_2(Q)$ is the projection operator of a function onto Q ; and the operators $R_{ijQ}, R_{iQ}: L_2(Q) \rightarrow L_2(Q)$ defined by the formulas $R_{ijQ} = P_Q R_{ij} I_Q$, $R_{iQ} = P_Q R_i I_Q$.

Let $H^1(Q)$ be Sobolev space of complex-valued functions from $L_2(Q)$ having all generalized derivatives of the first order from $L_2(Q)$.

We also introduce the sesquilinear form $a[v, w]$ in $L_2(Q)$ with domain $H^1(Q)$ by the formula

$$a[v, w] = \sum_{i,j=1}^n (R_{ijQ} v_{x_j}, w_{x_i})_{L_2(Q)} + \sum_{i=1}^n (R_{iQ} v_{x_i}, w)_{L_2(Q)} + (R_{0Q} v, w)_{L_2(Q)}.$$

The difference operators $R_{ijQ}, R_{iQ}, R_{0Q}: L_2(Q) \rightarrow L_2(Q)$ are bounded. Therefore, there exists a constant $c_0 > 0$ such that

$$|a[v, w]| \leq c_0 \|v\|_{H^1(Q)} \|w\|_{H^1(Q)} \quad (v, w \in H^1(Q)). \tag{2.2}$$

Since the sesquilinear form $a[v, w]$ is continuous with respect to w in $H^1(Q)$, there exists a linear bounded operator $A: H^1(Q) \rightarrow [H^1(Q)]' = \tilde{H}^{-1}(Q)$ such that

$$\langle Av, \bar{w} \rangle = a[v, w] \quad (v, w \in H^1(Q)), \quad (2.3)$$

where $\langle \cdot, \cdot \rangle$ denotes a dual pairing with respect to scalar product in $L_2(Q)$.

Definition 2.1 *The form $a[v, w]$ is said to be coercive if there exist numbers $c_1 > 0$ and $c_2 \geq 0$ such that*

$$\operatorname{Re} a[v, v] \geq c_1 \|v\|_{H^1(Q)}^2 - c_2 \|v\|_{L_2(Q)}^2 \quad (v \in H^1(Q)). \quad (2.4)$$

For the necessary and sufficient conditions of coercivity in algebraic form see [10, Lemma 2.2] (it is true also in the case ∂Q is Lipschitz, only the second term there must be written as $k_2 \|v\|_{L_2(Q)}^2$). Further, we shall assume that the form $a[v, w]$ is coercive.

We can assume that $c_2 = 0$ in inequality (2.4). In opposite case, we set $u = z e^{c_2 t}$ that transforms operator A to $A + c_2 I$.

To formulate the definition of the weak solution of problem (1.1)–(1.3) we introduce the following space

$$W_p(A) = \left\{ u \in L_p(0, T; H^1(Q)) : u' \in L_p(0, T; \tilde{H}^{-1}(Q)) \right\} \quad (1 < p < \infty).$$

Definition 2.2 *The function $u \in W_p(A)$ is said to be a weak solution of problem (1.1)–(1.3) if it satisfies equation*

$$\frac{du}{dt} + Au = f \text{ for almost all } t \in (0, T)$$

and initial condition

$$u|_{t=0} = \varphi.$$

Note that from [4, Ch. XVIII, §1, Sec. 4, Proposition 9] it follows that for function $u \in W_p(A)$, the value $u(0) \in \tilde{H}^{-1}(Q)$ is defined (the distributions should be taken on $(0, T)$).

To formulate the theorem about solvability of problem (1.1)–(1.3) we need the notation of Besov spaces. We shall introduce the smoothness modul of the second order for a function $f \in L_p(Q)$:

$$\omega_p^2(f, t)_p = \sup_{|h| \leq t} \left\| (f(x-h) - f(x))^2 \right\|_{L_p(Q)}.$$

Let $1 \leq \theta < \infty$ and $s > 0$ and $s = \tilde{s} + \alpha$, where $\tilde{s}$ is an integer and $0 < \alpha \leq 1$.

Definition 2.3 ([3]) *We say that function $f \in L_p(\mathbb{R}^n)$ subsets to functional space $B_{p,\theta}^s(\mathbb{R}^n)$ if it has partial unmixed derivatives up to the $\tilde{s}$ -th order integrated in the p -th power ($1 \leq p < \infty$) and*

$$\|f\|_{B_{p,\theta}^s(\mathbb{R}^n)} = \left(\|f\|_{W_{p,\theta}^{\tilde{s}}(\mathbb{R}^n)}^\theta + \int_0^\infty \left| \frac{\omega_p^2(f^{(\tilde{s})}, t)}{t^\alpha} \right|^\theta \frac{dt}{t} \right)^{1/\theta} < \infty.$$

The space $B_{p,\theta}^s(Q)$ is defined as a restriction of the space $B_{p,\theta}^s(\mathbb{R}^n)$ on Q with inf-norm (see [11, Sec. 4.2.1]). We set $B_{p,p}^0(Q) = L_p(Q)$ and $\tilde{B}_{p',\theta'}^s(Q)$ denotes the dual to $B_{p,\theta}^s(Q)$, where $1/p + 1/p' = 1/\theta + 1/\theta' = 1$.

Remark 2.1 We can use interpolation theorems for functions defined on $\mathbb{R}^n$ because Theorem 1 from Sec. 4.3.1 in [11] is true for Lipschitz domain as well by virtue of existing of extension operator independent on p , θ , and s (see [7]). Definitions of real and complex interpolations can be seen in [11].

The form $a[v, w]$ defines an unbounded linear operator $\mathcal{A}: D(\mathcal{A}) \rightarrow L_2(Q)$, where domain $D(\mathcal{A})$ consists from such $v \in H^1(Q)$ that there exists an element $\mathcal{A}v \in L_2(Q)$ such that

$$(\mathcal{A}v, w)_{L_2(Q)} = a[v, w] \quad (\forall w \in H^1(Q)).$$

By virtue of density $H^1(Q)$ in $L_2(Q)$, operator $\mathcal{A}$ is the part of A in $L_2(Q)$. The operator $\mathcal{A}$ is closed and the space $D(\mathcal{A})$ is a Hilbert space (with a graph norm) dense in $H^1(Q)$ (see Theorem 2.1 in [5, Ch. VI]).

Theorem 2.2 Let the form $a[v, w]$ be coercive with $c_2 = 0$, $1 < p < \infty$, and $f \in L_p(0, T; \tilde{H}^{-1}(Q))$. Then problem (1.1)–(1.3) has a unique weak solution $u \in W_p(A)$ if and only if $\varphi \in B_{2,p}^{1-\frac{2}{p}}(Q)$ for $p \geq 2$ and $\varphi \in \tilde{B}_{2,p}^{1-\frac{2}{p}}(Q)$ for $p < 2$. In fact, this solution is given by the formula

$$u(x, t) = T_t \varphi(x) + \int_0^t T_{t-s} f(x, s) ds, \quad (2.5)$$

where $\{T_t\}$ ($t \geq 0$) is the analytic semigroup generated by the operator $-A$.

Proof. By virtue of [6, Theorem 1.55], operator $-A$ is a generator of strongly continuous analytic semigroup T_t . Then the statement of the theorem follows from Theorems 3.6 and 3.7 in [2] for $\varphi \in (\tilde{H}^{-1}(Q), H^1(Q))_{1-\frac{1}{p}, p}$ (see [2, Ch. 1, Sec. 3]).

From [9, Corollary 3.3] it follows that $D(\mathcal{A}^{1/2}) = H^1(Q)$ that is equal to the identity $D(A^\alpha) = [\tilde{H}^{-1}(Q), H^1(Q)]_\alpha$ for $0 \leq \alpha \leq 1$ (see [1, Theorem 3.5]).

For $p \geq 2$, using formula (2) of Sec. 1.15.4 in [11], we obtain that

$$\begin{aligned} (\tilde{H}^{-1}(Q), H^1(Q))_{1-\frac{1}{p}, p} &= (D(A^0), D(A))_{1-\frac{1}{p}, p} = (D(A^{1/2}), D(A))_{1-\frac{2}{p}, p} \\ &= (L_2(Q), H^1(Q))_{1-\frac{2}{p}, p} = B_{2,p}^{1-\frac{2}{p}}(Q). \end{aligned}$$

For $p < 2$, using Theorem 2 of Sec. 1.10.3 and theorem on duality of Sec. 1.11.2 in [11], we obtain that

$$\begin{aligned} (\tilde{H}^{-1}(Q), H^1(Q))_{1-\frac{1}{p}, p} &= (H^1(Q), \tilde{H}^{-1}(Q))_{\frac{1}{p}, p} = (\tilde{H}^{-1}(Q), H^1(Q))'_{\frac{1}{p}, p'} \\ &= \left([\tilde{H}^{-1}(Q), H^1(Q)]_{1/2}, [\tilde{H}^{-1}(Q), H^1(Q)]_1 \right)'_{\frac{2}{p}-1, p'} \\ &= (L_2(Q), H^1(Q))'_{\frac{2}{p}-1, p'} = \left(B_{2,p'}^{\frac{2}{p}-1}(Q) \right)' \\ &= \tilde{B}_{2,p}^{1-\frac{2}{p}}(Q). \end{aligned}$$

■

Remark 2.3 By virtue of [10, Theorem 3.2], if $p = 2$ and $f \in L_2(Q_T)$, then Definition 2.2 is equal to the definition of weak solution by integral equality.

Remark 2.4 We have to define boundary condition (1.2) for function $u(\cdot, t) \in H^1(Q)$. We rewrite it in the form

$$T^+u = 0 \quad \text{on } \Gamma_T.$$

The value of operator $T^+ : H^1(Q) \rightarrow \tilde{H}^{-1/2}(\partial Q)$ (where $\tilde{H}^{-1/2}(\partial Q)$ denotes the dual to $H^{1/2}(\partial Q) = [L_2(\partial Q), H^1(\partial Q)]_{1/2}$) on function $u \in W_p(A)$ is defined by the Green formula

$$\langle f(\cdot, t), \bar{w} \rangle = \langle u_t, \bar{w} \rangle - \langle T^+u, \bar{w}|_{\partial Q} \rangle_{\Gamma} + a[u(\cdot, t), w], \quad w \in H^1(Q).$$

If the function u is smooth enough (that, as shown in [10], takes place near a smooth part of the boundary), then

$$T^+u = \sum_{ij=1}^n R_{ij,Q} u_{x_j} \cos(\nu, x_i).$$

3 Strong solvability

We introduce the Hilbert space $W_p(\mathcal{A}) = \{w \in L_p(0, T; D(\mathcal{A})) : w_t \in L_p(0, T; L_2(Q))\}$.

Definition 3.1 Function $u \in W_p(\mathcal{A})$ is called a strong solution of problem (1.1)–(1.3) if it satisfies equation

$$\frac{du}{dt} + \mathcal{A}u(\cdot, t) = f \text{ for almost all } t \in (0, T)$$

and initial condition

$$u|_{t=0} = \varphi.$$

Theorem 3.1 Let the form $a[v, w]$ be coercive with $c_2 = 0$ and $1 < p \leq 2$. Then, for any $f \in L_p(0, T; L_2(Q))$ and $\varphi \in B_{2,p}^{2-\frac{2}{p}}(Q)$, problem (1.1)–(1.3) has a unique strong solution defined by formula (2.5).

Proof. By virtue of [10, Theorem 4.1], operator $-\mathcal{A}$ is a generator of strongly continuous analytic semigroup $\mathcal{T}_t$, and $\mathcal{T}_t u = T_t u$ for $u \in L_2(Q)$ (see [6, Theorem 1.55]). Then the statement of the theorem follows from Theorems 3.6 and 3.7 in [2] for $\varphi \in (L_2(Q), D(\mathcal{A}))_{1-\frac{1}{p}, p}$ (see [2, Ch. 1, Sec. 3]).

By virtue of Theorem 2 of Sec. 1.10.3 in [11],

$$\begin{aligned} (L_2(Q), D(\mathcal{A}))_{1-\frac{1}{p}, p} &= (L_2(Q), [L_2(Q), D(\mathcal{A})]_{1/2})_{2-\frac{2}{p}, p} \\ &= (L_2(Q), H^1(Q))_{2-\frac{2}{p}, p} \\ &= B_{2,p}^{2-\frac{2}{p}}(Q). \end{aligned}$$

■

Remark 3.2 As it follows from the proof, the restriction $p \leq 2$ can be omitted, but we should consider the space $(L_2(Q), D(\mathcal{A}))_{1-\frac{1}{p}, p}$ instead of $(L_2(Q), H^1(Q))_{2-\frac{2}{p}, p}$. Also, in general (without assumption on smoothness of the solution), the domain $D(\mathcal{A})$ is unknown. Thus, we use the fact that $D(\mathcal{A}^{1/2}) = H^1(Q)$.

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References

- [1] M. S. Agranovich, A. M. Selitskii, Fractional powers of operators corresponding to coercitive problems in Lipschitz domains, *Funktsional'nyi Analiz i Ego Prilozheniya* 47 (2) (2013) 2–17, English transl. in: *Funct. Anal. Appl.* 47 (2) (2013) 83–95.
- [2] A. Ashyralyev, P. E. Sobolevskii, *Well-Posedness of Parabolic Difference Equations*, Birkhäuser, Basel, 1994.
- [3] O. V. Besov, Investigation of a class of function spaces in connection with imbedding and extension theorems, *Trudy Mat. Inst. Steklov.* 60 (1961) 42–81 (in Russian).
- [4] R. Dautray, J.-L. Lions, *Analyse Mathématique et Calcul Numérique t. 3.*, MASSON, Paris, 1985.
- [5] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1995.
- [6] E. M. Ouhabaz, *Analysis of Heat Equations on Domains*, Princeton Univ. Press, Princeton and Oxford, 2005.
- [7] V. S. Rychkov, On restrictions and extensions of the Besov and Triebel–Lizorkin spaces with respect to Lipschitz domains, *J. London Math. Soc.* 60 (2) (1997) 237–257.
- [8] A. M. Selitskii, The space of initial data of the second boundary-value problem for parabolic differential-difference equation, *Contemporary Analysis and Applied Mathematics* 1 (1) (2013) 34–41.
- [9] A. M. Selitskii, The space of initial data of the Robin boundary-value problem for parabolic differential-difference equations, *Contemporary Analysis and Applied Mathematics* 1 (2) (2013) 91–97.
- [10] A. M. Selitskii, A. L. Skubachevskii, The second boundary-value problem for parabolic differential-difference equations, *Trudy Seminara I.G. Petrovskogo.* 26 (2007) 323–346, English transl. in: *J. Math. Sci.* 143 (4) (2007) 3386–3400.
- [11] H. Triebel, *Interpolation Theory, Functional Spaces, Differential Operators*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.