



Inequalities via n -Times Differentiable *Quasi*-Convex Functions

n -Mertebeden Türevlenebilir *Quasi*-Konveks Fonksiyonlar Yardımıyla Eşitsizlikler

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Abstract

In this paper, some integral inequalities for n -times differentiable quasi-convex functions are established.

Keywords: Hermite-Hadamard inequality, Hölder inequality, Quasi-convex functions, Power-mean inequality

Öz

Bu çalışmada, n -mertebeden türevlenebilir *quasi*-konveks fonksiyonlar için yeni bazı integral eşitsizlikler elde edilmiştir.

Anahtar Kelimeler: Hermite-Hadamard eşitsizliği, Hölder eşitsizliği, *Quasi*-konveks fonksiyonlar, Power-mean eşitsizliği

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1. Introduction

A function $f: [a, b] \rightarrow \mathbb{R}$ is said to be *quasi*-convex function on $[a, b]$ if the inequality $f(\alpha x + (1 - \alpha)y) \leq \max\{f(x), f(y)\}$ holds for all $x, y \in [a, b]$ and $\alpha \in [0, 1]$. For some results about *quasi*-convexity see [1], [5], [7], [8], [13].

Hermite-Hadamard inequality is defined as below:

Let $f: I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on an interval I and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}$$

holds. Both inequalities hold in the reversed direction if f is concave. For some inequalities, generalizations and applications concerning Hermite-Hadamard inequality see [1], [3], [6], [7], [9]-[12], [16] and [17].

Recently, in the literature there are so many manuscripts about n -times differentiable functions on several kinds of convexities. In references [3], [4], [6], [9], [10] and [14]-[16] readers can find some results about this issue.

The main aim of this paper is to prove new integral inequalities for n -times differentiable *quasi*-convex functions.

To prove our main results, we use the following lemmas.

Lemma 1.1. [2] Let $f: [a, b] \rightarrow \mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}(n \geq 1)$ is absolutely continuous on $[a, b]$. Then for any $x \in [a, b]$ one has the equality:

$$\int_a^b f(t) dt = \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b) \right] + \frac{1}{n!} \int_a^b (x-t)^n f^{(n)}(t) dt.$$

Lemma 1.2. [6] Let $f: I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$, $a, b \in I^\circ$ with $a < b$, $f^{(n)}$ exists on I° and $f^{(n)} \in L(a, b)$ for $n \geq 1$. We have the following identity:

$$\begin{aligned} & \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a) \\ &= \frac{(b-a)^n}{2n!} \int_0^1 t^{n-1} (n-2t) f^{(n)}(ta + (1-t)b) dt. \end{aligned}$$

2. Inequalities for n -Times Differentiable *Quasi*-Convex Functions Via Lemma 1.1.

The first result is given in the following theorem:

Theorem 2.1. Let $f: [a, b] \rightarrow \mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}(n \geq 1)$ is absolutely continuous on $[a, b]$. If $|f^{(n)}|$ is *quasi*-convex on $[a, b]$, following inequality holds

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$$\left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b) \right] \right|$$

$$\leq \frac{1}{n!} \left\{ \left[\max\{|f^{(n)}(a)|, |f^{(n)}(x)|\} \right] \frac{(x-a)^{n+1}}{n+1} + \left[\max\{|f^{(n)}(x)|, |f^{(n)}(b)|\} \right] \frac{(b-x)^{n+1}}{n+1} \right\}$$

for all $x \in [a, b]$.

Proof. From Lemma 1.1., property of the modulus and *quasi*-convexity of $|f^{(n)}|$, we can write

$$\left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b) \right] \right|$$

$$\leq \frac{1}{n!} \int_a^b |x-t|^n |f^{(n)}(t)| dt$$

$$= \frac{1}{n!} \left\{ \int_a^x (x-t)^n |f^{(n)}(t)| dt + \int_x^b (t-x)^n |f^{(n)}(t)| dt \right\}$$

$$= \frac{1}{n!} \left\{ \int_a^x (x-t)^n \left| f^{(n)} \left(\frac{t-a}{x-a} x + \frac{x-t}{x-a} a \right) \right| dt \right.$$

$$+ \left. \int_x^b (t-x)^n \left| f^{(n)} \left(\frac{t-x}{b-x} b + \frac{b-t}{b-x} x \right) \right| dt \right\}$$

$$\leq \frac{1}{n!} \left\{ \int_a^x (x-t)^n [\max\{|f^{(n)}(a)|, |f^{(n)}(x)|\}] \right.$$

$$+ \left. \int_x^b (t-x)^n [\max\{|f^{(n)}(x)|, |f^{(n)}(b)|\}] dt \right\}.$$

If we use the fact that

$$\int_a^x (x-t)^n dt = \frac{(x-a)^{n+1}}{n+1}$$

and

$$\int_x^b (t-x)^n dt = \frac{(b-x)^{n+1}}{n+1}$$

we get the desired result.

Corollary 2.2. In Theorem 2.1., if we choose $x = \frac{a+b}{2}$, following inequality holds:

$$\left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{2} \right)^{k+1} [f^{(k)}(a) + (-1)^k f^{(k)}(b)] \right|$$

$$\leq \frac{(b-a)^{n+1}}{2^{n+1}(n+1)!} \left\{ \left[\max\left\{ \left| f^{(n)} \left(\frac{a+b}{2} \right) \right|, \left| f^{(n)} \left(\frac{a+b}{2} \right) \right| \right\} \right] + \left[\max\left\{ \left| f^{(n)} \left(\frac{a+b}{2} \right) \right|, \left| f^{(n)}(b) \right| \right\} \right] \right\}.$$

Theorem 2.3. Let $f: [a, b] \rightarrow \mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}$ ($n \geq 1$) is absolutely continuous on $[a, b]$. If $|f^{(n)}|^q$ is *quasi*-convex on $[a, b]$, following inequality holds for all $x \in [a, b]$

$$\left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b) \right] \right|$$

$$\leq \frac{(b-a)^{1/q}}{n!} \left(\frac{(x-a)^{np+1} + (b-x)^{np+1}}{np+1} \right)^{1/p} \left[\max \left\{ \left| f^{(n)}(a) \right|^q, \left| f^{(n)}(b) \right|^q \right\} \right]^{1/q}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 1.1., property of the modulus, well-known Hölder integral inequality and *quasi*-convexity of $|f^{(n)}|^q$, we can write

$$\left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b) \right] \right|$$

$$\leq \frac{1}{n!} \left(\int_a^b |x-t|^{np} dt \right)^{1/p} \left(\int_a^b |f^{(n)}(t)|^q dt \right)^{1/q}$$

$$= \frac{1}{n!} \left(\int_a^x (x-t)^{np} dt + \int_x^b (t-x)^{np} dt \right)^{1/p} \left(\int_a^b \left| f^{(n)} \left(\frac{b-t}{b-a} a + \frac{t-a}{b-a} b \right) \right|^q dt \right)^{1/q}$$

$$\leq \frac{1}{n!} \left(\frac{(x-a)^{np+1} + (b-x)^{np+1}}{np+1} \right)^{1/p} \left(\int_a^b \left[\max \left\{ \left| f^{(n)}(a) \right|^q, \left| f^{(n)}(b) \right|^q \right\} \right] dt \right)^{1/q}$$

$$= \frac{(b-a)^{1/q}}{n!} \left(\frac{(x-a)^{np+1} + (b-x)^{np+1}}{np+1} \right)^{1/p} \left[\max \left\{ \left| f^{(n)}(a) \right|^q, \left| f^{(n)}(b) \right|^q \right\} \right]^{1/q}$$

The proof is completed.

3. Inequalities for n -Times Differentiable *Quasi*-Convex Functions Via Lemma 1.2.

Theorem 3.1. Let $f: [a, b] \rightarrow \mathbb{R}$ be an n -times differentiable function. If $|f^{(n)}|^q$ is *quasi*-convex on $[a, b]$ following inequality holds

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a) \right|$$

$$\leq \frac{(b-a)^n (n-1)}{2(n+1)!} [\max\{|f^{(n)}(a)|^q, |f^{(n)}(b)|^q\}]^{1/q}$$

for $q \geq 1$ and $n \geq 2$.

Proof. From Lemma 1.2., property of the modulus, well-known power-mean integral inequality and *quasi*-convexity of $|f^{(n)}|^q$ we can write

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a) \right|$$

$$\leq \frac{(b-a)^n}{2n!} \int_0^1 t^{n-1} (n-2t) |f^{(n)}(ta + (1-t)b)| dt$$

$$\leq \frac{(b-a)^n}{2n!} \left(\int_0^1 t^{n-1} (n-2t) dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t^{n-1} (n-2t) \left| f^{(n)}(ta + (1-t)b) \right|^q dt \right)^{1/q}$$

$$\leq \frac{(b-a)^n}{2n!} \left(\frac{n-1}{n+1} \right)^{1-\frac{1}{q}} \left(\int_0^1 t^{n-1} (n-2t) \left[\max \left\{ \left| f^{(n)}(a) \right|^q, \left| f^{(n)}(b) \right|^q \right\} \right] dt \right)^{1/q}$$

$$= \frac{(b-a)^n (n-1)}{2(n+1)!} [\max\{|f^{(n)}(a)|^q, |f^{(n)}(b)|^q\}]^{1/q}$$

which completes the proof.

Theorem 3.2. Let $f: [a, b] \rightarrow \mathbb{R}$ be an n -times differentiable function. If $|f^{(n)}|^q$ is *quasi*-convex on $[a, b]$ following inequality holds

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a) \right| \\ & \leq \frac{(b-a)^n}{2^{1+\frac{1}{q}} n!} \left(\frac{1}{np-p+1} \right)^{\frac{1}{p}} \left(\frac{n^{q+1} - (n-2)^{q+1}}{q+1} \right)^{\frac{1}{q}} \\ & \times [\max\{|f^{(n)}(a)|^q, |f^{(n)}(b)|^q\}]^{\frac{1}{q}} \\ & \text{for } n \geq 2, q > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1. \end{aligned}$$

Proof. From Lemma 1.2., property of the modulus, well-known Hölder integral inequality and *quasi*-convexity of $|f^{(n)}|^q$ we can write

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a) \right| \\ & \leq \frac{(b-a)^n}{2n!} \int_0^1 t^{n-1} (n-2t) |f^{(n)}(ta + (1-t)b)| dt \\ & \leq \frac{(b-a)^n}{2n!} \left(\int_0^1 t^{(n-1)p} dt \right)^{\frac{1}{p}} \left(\int_0^1 (n-2t)^q \left| \frac{f^{(n)}(ta + (1-t)b)}{(1-t)b} \right|^q dt \right)^{\frac{1}{q}} \\ & \leq \frac{(b-a)^n}{2n!} \left(\frac{1}{np-p+1} \right)^{\frac{1}{p}} \left(\int_0^1 (n-2t)^q \left[\max\left\{ \left| \frac{f^{(n)}(a)}{(1-t)b} \right|^q, \left| \frac{f^{(n)}(b)}{(1-t)b} \right|^q \right\} \right] dt \right)^{\frac{1}{q}} \\ & = \frac{(b-a)^n}{2^{1+\frac{1}{q}} n!} \left(\frac{1}{np-p+1} \right)^{\frac{1}{p}} \left(\frac{n^{q+1} - (n-2)^{q+1}}{q+1} \right)^{\frac{1}{q}} \\ & \times [\max\{|f^{(n)}(a)|^q, |f^{(n)}(b)|^q\}]^{\frac{1}{q}}. \end{aligned}$$

The proof is completed.

4. References

- Alomari, M., Darus, M., Kirmacı U.S. 2010.** Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means, *Comput. Math. Applic.*, 59: 225-232.
- Barnett, N. S., Dragomir, S. S. 2002.** Applications of Ostrowski's version of the Grüss inequality for trapezoid type rules, *RGMA Res. Rep. Coll.*, 5.
- Bai, S.-P., Wang, S.-H., Qi, F. 2012.** Some Hermite-Hadamard type inequalities for n -time differentiable (α, m) -convex functions, *J. Inequal. Appl.*, 2012:267.
- Cerone, P., Dragomir, S. S., Roumelotis, J. 1999.** Some Ostrowski type inequalities for n -time differentiable mappings and applications, *Demonstr. Math.*, 32: 697-712.
- Dragomir, S. S., Pearce C. E. M. 2012.** Jensen's inequality for quasi-convex functions, *NACO*, 2: 279-291.
- Hwang, D.-Y. 2003.** Some inequalities for n -time differentiable mappings and applications, *Kyungpook Math. J.*, 43: 335-343.
- Hussain, S., Qaisar, S. 2013.** New Integral Inequalities of the Type of Hermite-Hadamard Through Quasi Convexity, *Punjab University J. of Math.*, 45.
- Ion, D. A. 2007.** Some estimates on the Hermite-Hadamard inequality through quasi-convex functions, *An. Univ. Craiova, Ser. Mat. Inf.*, 34: 83-88.
- Jiang, W.-D., Niu, D.-W., Hua, Y., Qi, F. 2012.** Generalizations of Hermite-Hadamard inequality to n -time differentiable functions which are quasi-convex in the second sense, *Analysis (Munich)*, 32: 209-220.
- Kechriniotis, A. I., Theodorou, Y. A. 2008.** New integral inequalities for n -time differentiable functions with applications for pdfs, *Appl. Math. Sciences*, 2: 353 - 362.
- Kavurmacı, H., Avcı, M., Özdemir, M. E. 2011.** New inequalities of Hermite-Hadamard type for convex functions with applications, *J. Inequal. Appl.*, 2011:86.
- Özdemir, M. E., Kavurmacı, H., Akdemir, A. O., Avcı, M. 2012.** Inequalities for convex and s -convex functions on $\Delta=[a,b] \times [c,d]$, *J. Inequal. Appl.*, 2012:20.
- Özdemir, M. E., Yıldız, Ç., Akdemir, A. O. 2012.** On some new Hadamard-type inequalities for co-ordinated quasi-convex functions, *Hacettepe J. Math. Statistics*, 41: 697 - 707.
- Pachpatte, B. G., 2004.** New inequalities of Ostrowski and Trapezoid type for n -time differentiable functions, *Bull. Korean Math. Soc.* 41: 633-639.
- Sofa, A. 2002.** Integral inequalities for n -times differentiable mappings, with multiple branches, on the $L_{\{p\}}$ norm, *Soochow J. Math.*, 28: 179-221
- Wang, S.-H., Xi, B.-Y., Qi, F. 2012.** Some new inequalities of Hermite-Hadamard type for n -time differentiable functions which are m -convex, *Analysis*, 32: 247-262.
- Xi, B.-Y., Qi, F. 2013.** Hermite-Hadamard type inequalities for functions whose derivatives are of convexities, *Nonlinear Funct. Anal. Appl.*, 18: 163-176.