

Concircularly and Conharmonically Flat Warped-Twisted Product Manifolds with Applications

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ABSTRACT

In this paper, we investigate warped-twisted product manifolds that are both concircularly flat and conharmonically flat. We examine the structure of their factor manifolds under these curvature conditions. Then we characterize the factor manifolds of a warped-twisted product manifold in terms of Ricci soliton and Yamabe soliton structures. Moreover, we introduce the notions of a warped-twisted generalized Robertson-Walker spacetime and a warped-twisted standard static spacetime. We also provide some applications for concircularly flat warped-twisted product manifolds that admit these spacetime metric structures. Finally, we prove that one of the factor manifolds of these spacetimes is gradient almost ζ -Yamabe soliton.

Mathematics Subject Classification (2020): 53C25, 53C20, 53Z05

Keywords: Warped product, twisted product, concircularly flat manifold, conharmonically flat manifold, Ricci soliton, generalized Robertson–Walker spacetime.

1. INTRODUCTION

The concircular curvature tensor C is invariant under concircular transformation. A concircular transformation preserves geodesic circles (Yano (1940)). The tensor C measures the deviation of the Riemannian manifold from constant curvature (Yano (1940)), since a concircularly flat manifold has a constant curvature. If the Laplacian of a function vanishes then it is a harmonic function and the conharmonic transformation preserves the harmonicity of a certain function. The conharmonic transformation keeps the conharmonic curvature tensor $\mathcal{H}$ invariant.

The notion of a warped product which is a complete manifold of negative curvature was presented by Bishop and O’Neill (1969). The concept of warped-twisted product (Gerdan Aydın and Taştan (2022); Taştan and Gerdan (2018); Taştan and Gerdan Aydın (2022)), doubly warped product (Gutierrez and Olea (2012)), twisted product (Ponge and Reckziegel (1993)) and warped product (Chen (2017)) are the special types of the doubly twisted products (Ponge and Reckziegel (1993); Yano (1940)). Ricci solitons (Hamilton (1988)) and Yamabe solitons (Chen and Deshmukh (2018)) have recently been investigated by many authors. Ricci solitons are in close relationship with the warped product manifolds. Cao et al. (2012) showed that a gradient Yamabe soliton has a warped product structure. Besides that, some authors considered gradient solitons on doubly warped and twisted product manifolds (Blaga (2022); Güler and Taştan (2022)). Some applications of twisted products are also investigated by Mantica and Molinari (2017).

Moreover, Chen (2015) studied Ricci solitons when their potential vector field is concircular. De et al. (2020) investigated concircularly flat warped product manifolds and concircular curvature tensor when it is divergence-free on warped product manifolds. Ahsan and Siddiqui (2009) considered perfect fluid spacetimes with the concircular curvature tensor which is either vanishing or divergence-free. The twisted product manifold was investigated by Agaoka et al. (1999) when its conharmonic curvature tensor vanishes. Agaoka et al. (2000) also showed that a Riemannian manifold is locally a product manifold under certain conditions, provided that it is conharmonically flat. Besides these studies, an application of the conharmonic curvature tensor satisfying the Einstein field equations on a four dimensional spacetime was given by Siddiqui and Ahsan (2010). In the present paper, we provide the basic background of warped-twisted product manifolds and the definitions of gradient solitons, concircularly and conharmonically flatness in Section 2. In Section 3, we investigate the structure of factor manifolds when the warped-twisted product manifold is both concircularly flat and conharmonically flat. Moreover, we introduce warped-twisted

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Submitted: 09.09.2025 • **Revision Requested:** 03.09.2025 • **Last Revision Received:** 24.09.2025 • **Accepted:** 30.10.2025



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generalized Robertson–Walker spacetime and warped-twisted standard static spacetime and we give some results when these spacetime models are concircularly flat in Section 4.

2. PRELIMINARIES

2.1. Warped-twisted products

Let $f_2 : N_2 \rightarrow (0, \infty)$ and $f_1 : N_1 \times N_2 \rightarrow (0, \infty)$ be smooth functions. Then ${}_{f_2}N_1 \times_{f_1} N_2$ is called a *warped-twisted product manifold* (Taştan and Gerdan (2018)) with warping function f_2 and twisting function f_1 when it has a metric tensor g defined by

$$g := (f_2 \circ \pi_2)^2 \pi_1^*(g_1) + f_1^2 \pi_2^*(g_2), \quad (1)$$

where π_i denotes the canonical projection from the product $N_1 \times N_2$ to N_i , for $i = 1, 2$. If f_1 depends only on the points of N_1 , then ${}_{f_2}N_1 \times_{f_1} N_2$ becomes a *doubly warped product manifold* (Gutierrez and Olea (2012)) with warping functions f_i . If f_1 depends only on the points of N_2 , then ${}_{f_2}N_1 \times_{f_1} N_2$ becomes a *base conformal warped product manifold* (Dobarro and Ünal (2008)). A warped-twisted product is *non-trivial* except for doubly warped product and base conformal warped product case.

Let ${}_{f_2}N_1 \times_{f_1} N_2$ be a warped-twisted product manifold. Then, the following formulas hold:

$$\nabla_E F = \nabla_E^1 F - g(E, F) \nabla \ln(f_2 \circ \pi_2), \quad (2)$$

$$\nabla_L E = \nabla_E L = L(\ln(f_2 \circ \pi_2))E + E(\ln f_1)L, \quad (3)$$

$$\nabla_K L = \nabla_K^2 L + K(\ln f_1)L + L(\ln f_1)K - g(K, L) \nabla \ln f_1, \quad (4)$$

where $E, F \in \mathcal{L}(N_1)$ and $K, L \in \mathcal{L}(N_2)$. Here, ∇ is the Levi–Civita connection of g given in (1) and for $i \in \{1, 2\}$, ∇^i is the Levi–Civita connection of N_i .

Remark 2.1. We use the notation $p = \ln f_1$ (resp. $r = \ln f_2$) and we denote the function p (resp. r) and its pullback $p \circ \pi_1$ (resp. $r \circ \pi_2$) with the same notation. And also the set of lifts of vector fields are denoted by $\mathcal{L}(N_i)$ on N_i . Moreover, the same notation can be used for both a vector field and its lift and for both a connection on N_i and its pullback via π_i , since π_i is a positive homothety.

For any $E \in \mathcal{L}(N_1)$ and $K \in \mathcal{L}(N_2)$, we have from (2) and (3) that

$$h^\psi(E, K) = EK(\psi) - E(p)K(\psi) - E(\psi)K(r), \quad (5)$$

where ψ is a smooth function on the warped-twisted product ${}_{f_2}N_1 \times_{f_1} N_2$.

Next, we define $h_1^\psi(E, F) := EF(\psi) - (\nabla_E^1 F)(\psi)$, for all $E, F \in \mathcal{L}(N_1)$ and $h_2^\psi(K, L) := KL(\psi) - (\nabla_K^2 L)(\psi)$, for all $K, L \in \mathcal{L}(N_2)$. From (2) and (4), we have

$$h^\psi(E, F) = h_1^\psi(E, F) + g(E, F)g(\nabla r, \nabla \psi) \quad (6)$$

and

$$h^\psi(K, L) = h_2^\psi(K, L) - K(p)L(\psi) - L(p)K(\psi) + g(K, L)g(\nabla p, \nabla \psi). \quad (7)$$

Since $\nabla r \in \mathcal{L}(N_2)$, from (6) and (7) we deduce that

$$h^p(E, F) = h_1^p(E, F) + g(E, F)g(\nabla r, \nabla p), \quad (8)$$

$$h^r(E, F) = g(E, F)g(\nabla r, \nabla r), \quad (9)$$

$$h^p(K, L) = h_2^p(K, L) - K(p)L(p) - L(p)K(p) + g(K, L)g(\nabla p, \nabla p), \quad (10)$$

$$h^r(K, L) = h_2^r(K, L) - K(p)L(r) - L(p)K(r) + g(K, L)g(\nabla p, \nabla r). \quad (11)$$

Let ${}_{f_2}N_1 \times_{f_1} N_2$ be a warped-twisted product with the Riemann curvature tensor R . Then, from (2)–(4), we get:

Lemma 2.2. Let ${}^1R$ and ${}^2R$ be the lifts of the Riemann curvature tensors of (N_1, g_1) and (N_2, g_2) respectively. For $E, F, Z \in \mathcal{L}(N_1)$

and $K, L, W \in \mathcal{L}(N_2)$, we have

$$R_{EF}Z = {}^1R_{EF}Z + H^r(F)g(E, Z) - H^r(E)g(F, Z), \quad (12)$$

$$R_{EF}K = K(r)\left\{F(p)E - E(p)F\right\}, \quad (13)$$

$$R_{KL}E = E(p)\left\{L(r)K - K(r)L\right\} + KE(p)L - LE(p)K, \quad (14)$$

$$\begin{aligned} R_{EK}F &= \left\{h_1^p(E, F) + E(p)F(p)\right\}K + F(p)K(r)E \\ &\quad + g(E, F)\left\{H^r(K) + K(r)\nabla r\right\}, \end{aligned} \quad (15)$$

$$\begin{aligned} R_{KE}L &= \left\{h_2^r(K, L) + K(r)L(r) - K(r)L(p) - K(p)L(r)\right\}E \\ &\quad + \left\{L(r)E(p) - EL(p)\right\}K + \left\{E(p)\nabla p + H^p(E)\right\}g(K, L), \end{aligned} \quad (16)$$

$$\begin{aligned} R_{KL}W &= {}^2R_{KL}W - \left\{h_2^p(L, W) - W(p)L(p)\right\}K \\ &\quad + \left\{h_2^p(K, W) - W(p)K(p)\right\}L - \left\{H^p(K) + K(p)\nabla p\right\}g(L, W) \\ &\quad + \left\{H^p(L) + L(p)\nabla p\right\}g(K, W), \end{aligned} \quad (17)$$

where $H^f(E) := \nabla_E \nabla f$ is the Hessian tensor of a smooth function f on ${}_{f_2}N_1 \times_{f_1} N_2$.

Let ${}_{f_2}N_1 \times_{f_1} N_2$ be a warped-twisted product with the Ricci curvature tensor Ric . Then from (12)–(17) and (9)–(10), we get:

Lemma 2.3. Let ${}^1\text{Ric}$ and ${}^2\text{Ric}$ be the lifts of the Ricci curvature tensors of (N_1, g_1) and (N_2, g_2) respectively. For $E, F \in \mathcal{L}(N_1)$ and $K, L \in \mathcal{L}(N_2)$, we have

$$\begin{aligned} \text{Ric}(E, F) &= {}^1\text{Ric}(E, F) + h^r(E, F) - n_2\left\{h_1^p(E, F) + E(p)F(p)\right\} \\ &\quad - g(E, F)\left\{\Delta r + g(\nabla r, \nabla r)\right\}, \end{aligned} \quad (18)$$

$$\text{Ric}(E, K) = (1 - n_2)EK(p) + (n_1 + n_2 - 2)E(p)K(r), \quad (19)$$

$$\begin{aligned} \text{Ric}(K, L) &= {}^2\text{Ric}(K, L) + h^p(K, L) + (1 - n_2)h_2^p(K, L) + n_2K(p)L(p) \\ &\quad - g(K, L)\left\{\Delta p + g(\nabla p, \nabla p)\right\} \\ &\quad - n_1\left\{h_2^r(K, L) + K(r)L(r) - K(r)L(p) - K(p)L(r)\right\}, \end{aligned} \quad (20)$$

where Δ is the Laplacian operator on ${}_{f_2}N_1 \times_{f_1} N_2$ and $n_i = \dim(N_i)$, for $i \in \{1, 2\}$.

Let ${}_{f_2}N_1 \times_{f_1} N_2$ be a warped-twisted product with the Ricci operator Q . We give the following lemma from (18) and (20).

Lemma 2.4. Let ${}^1Q$ and ${}^2Q$ be the lifts of the Ricci operators of (N_1, g_1) and (N_2, g_2) respectively. For $E \in \mathcal{L}(N_1)$ and $K \in \mathcal{L}(N_2)$, we have:

$$QE = \frac{1}{f_2^2} \left({}^1QE - \frac{n_2}{f_1} H^{f_1}(E) - f_2^2 \Delta r \cdot E \right), \quad (21)$$

$$\begin{aligned} QK &= \frac{1}{f_1^2} \left({}^2QK + \frac{(2 - n_2)}{f_1} H_2^{f_1}(K) + 2(n_2 - 2) \frac{K(f_1)\nabla f_1}{f_1^2} \right. \\ &\quad \left. - n_1 \left\{ \frac{1}{f_2} H_2^{f_2}(K) - \frac{K(f_2)\nabla f_1 + K(f_1)\nabla f_2}{f_1 f_2} \right\} - f_1^2 \Delta p \cdot K \right). \end{aligned} \quad (22)$$

2.2. Gradient Yamabe/Ricci Solitons, Concircularly/Conharmonically Flat Manifolds

Let (N^n, g) be a pseudo-Riemannian manifold. Let ϕ be a smooth function on N and λ be a real constant satisfying

$$h^\phi = (\tau - \lambda)g. \quad (23)$$

Then N is a *gradient Yamabe soliton* (Chow (1992)), where τ is the scalar curvature of (N^n, g) and h^ϕ is the Hessian of the ϕ . Let γ be a smooth function on N . If $h^\phi = \gamma g$, then (N^n, g, ϕ) is a *conformal gradient soliton*. On the other hand, if we have

$$h^\phi + \text{Ric} = \lambda g, \quad (24)$$

then (N^n, g) is a *gradient Ricci soliton* (Hamilton (1988)), where ϕ is a smooth function on N , λ is a real constant and Ric is the Ricci curvature of (N^n, g) . Moreover, if we have

$$h^\phi = (\tau - \lambda)g + \nu \zeta \otimes \zeta, \quad (25)$$

for a 1-form ζ and for a real constant ν on N , then (N^n, g) is a *gradient ζ -Yamabe soliton* (Chen and Deshmukh (2018)).

(N^n, g) is a *gradient ζ -Ricci soliton* (Cho and Kimura (2009)), if we have

$$h^\phi + \text{Ric} = \lambda g + \nu \zeta \otimes \zeta, \quad (26)$$

for two real constants λ and ν , and for a 1-form ζ on N .

The soliton is called shrinking, expanding or steady when $\lambda > 0$, $\lambda < 0$ or $\lambda = 0$ respectively. If λ is a smooth function on N , then (N^n, g, ϕ) is called an *almost soliton* in the above definitions.

(N^n, g) is called a *gradient f -almost Ricci soliton* (Gomes et al. (2017)) if the following equation holds

$$f h^\phi + \text{Ric} = \lambda g,$$

for some smooth functions f , ϕ and λ on N . Additionally, (N^n, g) is a *gradient f -almost ζ -Ricci soliton* if we have

$$f h^\phi + \text{Ric} = \lambda g + \nu \zeta \otimes \zeta.$$

If the condition $\text{Ric} = \frac{\tau}{n}g$ holds for $n \geq 2$, then (N^n, g) is called an *Einstein manifold* (Besse (1987)).

Let us give the definition of concircular and conharmonic curvature tensors. Let

$$C = R - \frac{\tau}{n(n-1)}G, \quad (27)$$

where $G = \frac{1}{2}(g \wedge g)$ for the Kulkarni-Nomizu product $\wedge$ and R is the Riemann curvature tensor of N . Then, C is called the *concircular curvature tensor* (Yano (1940)). We say that N is *concircularly flat* when C vanishes identically on N . Let

$$\mathcal{H}_{EF}Z = R_{EF}Z - \frac{1}{n-2} \{g(F, Z)QE - g(E, Z)QF + \text{Ric}(F, Z)E - \text{Ric}(E, Z)F\}$$

for every vector fields E, F, Z on N . Then $\mathcal{H}$ is called the *conharmonic curvature tensor* (Ishii (1957)). We say that N is *conharmonically flat* when $\mathcal{H}$ vanishes identically on N .

3. MAIN RESULTS

In this section, we obtain some results when a warped-twisted product manifold is concircularly and conharmonically flat.

Lemma 3.1. *Let $f_2 N_1 \times_{f_1} N_2$ be a warped-twisted product manifold. Then the following equations hold for any $E, F, Z \in \mathcal{L}(N_1)$ and $K, L, W \in \mathcal{L}(N_2)$:*

$$\begin{aligned} C_{EF}Z &= R_{EF}Z + g(E, Z)H^r(F) - g(F, Z)H^r(E) \\ &\quad - \frac{f_2^2 \tau}{2n(n-1)} \{g_1(F, Z)E - g_1(E, Z)F\}, \end{aligned} \quad (28)$$

$$C_{EF}K = K(r) \{F(p)E - E(p)F\}, \quad (29)$$

$$C_{KL}E = E(p) \{L(r)K - K(r)L\} + KE(p)L - LE(p)K, \quad (30)$$

$$\begin{aligned} C_{EK}F &= \{h_1^p(E, F) + E(p)F(p)\}K + F(p)K(r)E \\ &\quad + g(E, F) \{H^r(K) + K(r)\nabla r\} + \frac{f_2^2 \tau}{2n(n-1)} g_1(E, F)K, \end{aligned} \quad (31)$$

$$\begin{aligned} C_{KE}L &= \{h_2^r(K, L) + K(r)L(r) - K(r)L(p) - K(p)L(r)\}E \\ &+ \{L(r)E(p) - EL(p)\}K \\ &+ g(K, L)\{H^p(E) + E(p)\nabla p\} + \frac{f_1^2\tau}{2n(n-1)}g_2(K, L)E, \end{aligned} \quad (32)$$

$$\begin{aligned} C_{KL}W &= {}^2R_{KL}W - \{h_2^p(L, W) - W(p)L(p)\}K + \{h_2^p(K, W) - W(p)K(p)\}L \\ &+ g(K, W)\{H^p(L) + L(p)\nabla p\} - g(L, W)\{H^p(K) + K(p)\nabla p\} \\ &- \frac{f_1^2\tau}{2n(n-1)}\{g_2(L, W)K - g_2(K, W)L\}. \end{aligned} \quad (33)$$

Theorem 3.2. Let $f_2 N_1 \times_{f_1} N_2$ be a warped-twisted product manifold with $n_1, n_2 > 2$. If N is concircularly flat, then

(a) (N_1, g_1) is Einstein with ${}^1\text{Ric} = \nu_1 g_1$,

where $\nu_1 = f_2^2(n_1 - 1)\left\{g(\nabla r, \nabla r) + \frac{\tau}{2n(n-1)}\right\}$,

(b) (N_2, g_2) is a gradient ζ -Ricci soliton with ${}^2\text{Ric} + f h_2^p = \lambda g_2 + \nu_2 \zeta \otimes \zeta$,

where $f = 2 - n_2$, $\lambda = \Delta^2 p + \{f_1^2(n_2 - 1) - 1\}g(\nabla p, \nabla p) + \frac{f_1^2\tau}{2n(n-1)}(n_2 - 1)$, $\nu_2 = 3 - n_2 - f_1^2$ and $\zeta = \tilde{d}p$, where $\tilde{d}p$ is the pullback of dp to N_2 ,

(c) $N = N_1 \times_{f_1} N_2$ is a twisted product manifold or $G_{EF}\nabla p = 0$,

(d) $N =_{f_2} N_1 \times N_2$ is a warped product manifold or $G_{KL}\nabla r = 0$.

Proof. For any $E, F, Z \in \mathcal{L}(N_1)$,

$${}^1R_{EF}Z = -g(E, Z)H^r(F) + g(F, Z)H^r(E) + \frac{f_2^2\tau}{2n(n-1)}\{g_1(F, Z)E - g_1(E, Z)F\},$$

from (28). Hence, for any $T \in \mathcal{L}(N_1)$, using (1) and (9), it follows that

$$g_1(H^r(E), T) = g(\nabla r, \nabla r)g_1(E, T),$$

and thus, we obtain

$$\begin{aligned} g_1({}^1R_{EF}Z, T) &= -f_2^2g(\nabla r, \nabla r)g_1(E, Z)g_1(F, T) + f_2^2g(\nabla r, \nabla r)g_1(F, Z)g_1(E, T) \\ &+ \frac{f_2^2\tau}{2n(n-1)}\{g_1(F, Z)g_1(E, T) - g_1(E, Z)g_1(F, T)\}. \end{aligned}$$

Hence we get

$${}^1\text{Ric}(F, Z) = f_2^2(n_1 - 1)\left\{g(\nabla r, \nabla r) + \frac{\tau}{2n(n-1)}\right\}g_1(F, Z),$$

if we contract the previous equation over E and T . This proves (a). For any $K, L, W \in \mathcal{L}(N_2)$, we have

$$\begin{aligned} {}^2R_{KL}W &= \left\{h_2^p(L, W) - W(p)L(p)\right\}K - \left\{h_2^p(K, W) - W(p)K(p)\right\}L \\ &- g(K, W)\left\{H^p(L) + L(p)\nabla p\right\} + g(L, W)\left\{H^p(K) + K(p)\nabla p\right\} \\ &+ \frac{f_1^2\tau}{2n(n-1)}\{g_2(L, W)K - g_2(K, W)L\}, \end{aligned}$$

from (33). Hence, for any $S \in \mathcal{L}(N_2)$, using (1) and (10), it can be seen that

$$g_2(H^p(K), S) = h_2^p(K, S) - K(p)S(p) - S(p)K(p) + g(\nabla p, \nabla p)g_1(K, S),$$

and therefore, we obtain

$$\begin{aligned}
 g_2(^2R_{KL}W, S) = & \left\{ h_2^P(L, W) - W(p)L(p) \right\} g_2(K, S) \\
 & - \left\{ h_2^P(K, W) - W(p)K(p) \right\} g_2(L, S) \\
 & - g_2(K, W) \left\{ h_2^P(L, S) - 2L(p)S(p) + f_1^2 g_2(L, S)g(\nabla p, \nabla p) \right. \\
 & \left. + L(p)f_1^2 g_2(\nabla p, S) \right\} \\
 & + g_2(L, W) \left\{ h_2^P(K, S) - 2K(p)S(p) + f_1^2 g_2(K, S)g(\nabla p, \nabla p) \right. \\
 & \left. + K(p)f_1^2 g_2(\nabla p, S) \right\} \\
 & + \frac{f_1^2 \tau}{2n(n-1)} \{ g_2(L, W)g_2(K, S) - g_2(K, W)g_2(L, S) \}.
 \end{aligned}$$

By contracting this equation over K and S , we get

$$\begin{aligned}
 {}^2\text{Ric}(L, W) + (2 - n_2)h_2^P(L, W) = & \left\{ \Delta^2 p + \{ f_1^2(n_2 - 1) - 1 \} g(\nabla p, \nabla p) \right. \\
 & \left. + \frac{f_1^2 \tau}{2n(n-1)}(n_2 - 1) \right\} g_2(L, W) \\
 & + (3 - n_2 - f_1^2) \tilde{d}p(L) \tilde{d}p(W),
 \end{aligned}$$

which proves **(b)**. For any $E, F \in \mathcal{L}(N_1)$ and $K \in \mathcal{L}(N_2)$, we have

$$K(r) \left\{ F(p)E - E(p)F \right\} = 0,$$

from (29). It follows that $K(r) = 0$, for all $K \in \mathcal{L}(N_2)$, or $F(p)E - E(p)F = 0$, for every $E, F \in \mathcal{L}(N_1)$. From the last equation, we get

$$G_{EF} \nabla p = \frac{1}{2} \left\{ g(F, \nabla p)E - g(E, \nabla p)F \right\} = 0.$$

If $K(r) = 0$, we deduce that $N = N_1 \times_{f_1} N_2$ is a twisted product manifold, which completes the proof of **(c)**. We can prove **(d)** similarly from (30).

Lemma 3.3. *Let ${}_{f_2} N_1 \times_{f_1} N_2$ be a warped-twisted product manifold. Then, for any $E, F, Z \in \mathcal{L}(N_1)$ and $K, L, W \in \mathcal{L}(N_2)$, we have*

$$\begin{aligned}
 \mathcal{H}_{EF}Z = & {}^1R_{EF}Z + g(E, Z)H^r(F) - g(F, Z)H^r(E) \\
 & - \frac{1}{n-2} \left\{ \frac{1}{f_2^2} g(F, Z) \left\{ {}^1QE - \frac{n_2}{f_1} H_1^{f_1}(E) - f_2^2 \Delta r \cdot E \right\} \right. \\
 & - \frac{1}{f_2^2} g(E, Z) \left\{ {}^1QF - \frac{n_2}{f_1} H_1^{f_1}(F) - f_2^2 \Delta r \cdot F \right\} \\
 & + {}^1\text{Ric}(F, Z)E + h^r(F, Z)E - n_2 \{ h_1^P(F, Z)E + F(p)Z(p)E \} \\
 & - g(F, Z) \{ \Delta r + g(\nabla r, \nabla r) \} E \\
 & + {}^1\text{Ric}(E, Z)F + h^r(E, Z)F - n_2 \{ h_1^P(E, Z)F + E(p)Z(p)F \} \\
 & \left. - g(E, Z) \{ \Delta r + g(\nabla r, \nabla r) \} F \right\}, \tag{34}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{H}_{KL}W &= {}^2R_{KL}W - \{h_2^p(L, W) - W(p)L(p)\}K + \{h_2^p(K, W) - W(p)K(p)\}L \\
 &\quad - \{H^p(K) + K(p)\nabla p\}g(L, W) + \{H^p(L) + L(p)\nabla p\}g(K, W) \\
 &\quad - \frac{1}{n-2} \left\{ \frac{1}{f_1^2} g(L, W) \left({}^2QK + \frac{(2-n_2)}{f_1} H_2^{f_1}(K) + 2(n_2-2) \frac{K(f_1)\nabla f_1}{f_1^2} \right. \right. \\
 &\quad \left. \left. - n_1 \left(\frac{1}{f_2} H_2^{f_2}(K) - \frac{K(f_2)\nabla f_1 + K(f_1)\nabla f_2}{f_1 f_2} \right) - f_1^2 \Delta p \cdot K \right) \right. \\
 &\quad \left. - \frac{1}{f_1^2} g(K, W) \left({}^2QL + \frac{(2-n_2)}{f_1} H_2^{f_1}(L) + 2(n_2-2) \frac{L(f_1)\nabla f_1}{f_1^2} \right. \right. \\
 &\quad \left. \left. - n_1 \left(\frac{1}{f_2} H_2^{f_2}(L) - \frac{L(f_2)\nabla f_1 + L(f_1)\nabla f_2}{f_1 f_2} \right) - f_1^2 \Delta p \cdot L \right) \right. \\
 &\quad + \left({}^2\text{Ric}(L, W) + h^p(L, W) + (1-n_2)h_2^p(L, W) + n_2 L(p)W(p) \right. \\
 &\quad \left. - g(L, W)\{\Delta p + g(\nabla p, \nabla p)\} \right. \\
 &\quad \left. - n_1 \{h_2^r(L, W) + L(r)W(r) - L(r)W(p) - L(p)W(r)\} \right) K \\
 &\quad - \left({}^2\text{Ric}(K, W) + h^p(K, W) + (1-n_2)h_2^p(K, W) + n_2 K(p)W(p) \right. \\
 &\quad \left. - g(K, W)\{\Delta p + g(\nabla p, \nabla p)\} \right. \\
 &\quad \left. - n_1 \{h_2^r(K, W) + K(r)W(r) - K(r)W(p) - K(p)W(r)\} \right) L \Big\}. \tag{35}
 \end{aligned}$$

Theorem 3.4. Let $f_2 N_1 \times_{f_1} N_2$ be a conharmonically flat warped-twisted product manifold with $n_1, n_2 > 1$ and $n \neq n_1 + 2$. Then

(a) (N_1, g_1) is a gradient ζ -Ricci soliton, where

$$\lambda_1 = \frac{\{(n-2)(n_1-1) + n_1\}f_2^2 - n_1}{n - n_1 - 2} g(\nabla r, \nabla r) + \frac{1}{n - n_1 - 2} \left(\tau - \frac{n_2}{f_1} \Delta_1 f_1 - n_1(1 + f_2^2) \Delta r \right),$$

$$f = \frac{1}{(2 + n_1 - n)}, \quad \nu_1 = \frac{(-n_1 n_2 + f_2^2)}{n - n_1 - 2} \text{ and } \zeta = \hat{d}p, \text{ where } \hat{d}p \text{ is the pullback of } dp \text{ to } N_1,$$

(b) $h_2^{\varphi_2} + \frac{n_1}{n-2} {}^2\text{Ric} = \lambda_2 g_2 + \nu_2 dp \otimes dp + \xi_2(dp \otimes dr + dr \otimes dp)$,

$$\text{where } \varphi_2 = - \left\{ \left(\frac{3n_2 - n_2^2 - 1}{n-2} + n_2 - f_1^2 - 1 \right) p + \frac{2n_1 - 2}{n-2} + \frac{n_1 - n_1 n_2}{n-2} r \right\},$$

$$\begin{aligned}
 \lambda_2 &= \left\{ \frac{1}{n-2} \tau (f_1^4 n_2 + \frac{2(n_2-2)}{f_1^2(n-2)} - \frac{n_2^2 f_1^2}{n-2}) g(\nabla p, \nabla p) + \frac{2n_1}{(n-2)f_1^2} g(\nabla p, \nabla r) \right. \\
 &\quad \left. + (f_1^2 + \frac{2-n_2}{n-2}) \Delta_2 p - \frac{n_1}{n-2} \Delta_2 r + \frac{2f_1^2(1-n_2)}{n-2} \Delta p \right\},
 \end{aligned}$$

$$\nu_2 = 2 - n_2 - 2 \frac{n_2 - 2}{n-2} \text{ and } \xi_2 = - \frac{n_1 f_2}{n-2 f_1^3}.$$

Proof. For any $E, F, Z \in \mathcal{L}(N_1)$, we have

$$\begin{aligned}
 &{}^1R_{EF}Z + g(E, Z)H^r(F) - g(F, Z)H^r(E) \\
 &\quad - \frac{1}{n-2} \left\{ \frac{1}{f_2^2} g(F, Z) \left({}^1QE - \frac{n_2}{f_1} H_1^{f_1}(E) - f_2^2 \Delta r \cdot E \right) \right. \\
 &\quad \left. - \frac{1}{f_2^2} g(E, Z) \left({}^1QF - \frac{n_2}{f_1} H_1^{f_1}(F) - f_2^2 \Delta r \cdot F \right) \right. \\
 &\quad + {}^1\text{Ric}(F, Z)E + h^r(F, Z)E - n_2 \{h_1^p(F, Z)E + F(p)Z(p)E\} \\
 &\quad - g(F, Z)\{\Delta r + g(\nabla r, \nabla r)\}E \\
 &\quad + {}^1\text{Ric}(E, Z)F + h^r(E, Z)F - n_2 \{h_1^p(E, Z)F + E(p)Z(p)F\} \\
 &\quad \left. - g(E, Z)\{\Delta r + g(\nabla r, \nabla r)\}F \right\} = 0
 \end{aligned}$$

from (34). Hence, for any $T \in \mathcal{L}(N_1)$, using (1) and (9), it follows that

$$g_1(H^r(E), T) = g(\nabla r, \nabla r)g_1(E, T),$$

and thus, we obtain

$$\begin{aligned}
g_1({}^1R_{EF}Z, T) = & -g(E, Z)g_1(H^r(F), T) + g(F, Z)g_1(H^r(E), T) \\
& + \frac{1}{n-2} \left\{ \frac{1}{f_2^2} g(F, Z) \left(g_1({}^1QE, T) - \frac{n_2}{f_1} g_1(H_1^{f_1}(E), T) - f_2^2 \Delta r g_1(E, T) \right) \right. \\
& - \frac{1}{f_2^2} g(E, Z) \left(g_1({}^1QF, T) - \frac{n_2}{f_1} g_1(H_1^{f_1}(F), T) - f_2^2 \Delta r g_1(F, T) \right) \\
& + \left\{ {}^1\text{Ric}(F, Z) + h^r(F, Z) - n_2 \{ h_1^p(F, Z) + F(p)Z(p) \} \right. \\
& - g(F, Z) \{ \Delta r + g(\nabla r, \nabla r) \} \} g_1(E, T) \\
& + \left\{ {}^1\text{Ric}(E, Z) + h^r(E, Z) - n_2 \{ h_1^p(E, Z) + E(p)Z(p) \} \right. \\
& \left. \left. - g(E, Z) \{ \Delta r + g(\nabla r, \nabla r) \} \} g_1(F, T) \right\}.
\end{aligned}$$

Hence we get

$$\begin{aligned}
{}^1\text{Ric}(F, Z) + \frac{1}{(2+n_1-n)} h_1^{\varphi_1} = & \left\{ \left(\frac{\{(n-2)(n_1-1)+n_1\}f_2^2-n_1}{n-n_1-2} \right) g(\nabla r, \nabla r) \right. \\
& + \frac{1}{n-n_1-2} \left({}^1\tau - \frac{n_2}{f_1} \Delta_1 f_1 \right. \\
& \left. \left. - n_1(1+f_2^2)\Delta r \right) \right\} g_1(F, Z) \\
& + \frac{(-n_1n_2+f_2^2)}{n-n_1-2} \hat{d}p(F)\hat{d}p(Z),
\end{aligned}$$

if we contract the previous equation over E and T , where $\varphi_1 = (n_2 - n_1n_2p - n_2p)$. Thus, we obtain the assertion (a). For any $K, L, W \in \mathcal{L}(N_2)$, we have

$$\begin{aligned}
{}^2R_{KL}W = & \{h_2^p(L, W) - W(p)L(p)\}K - \{h_2^p(K, W) - W(p)K(p)\}L \\
& + \{H^p(K) + K(p)\nabla p\}g(L, W) - \{H^p(L) + L(p)\nabla p\}g(K, W), \\
& + \frac{1}{n-2} \left\{ \frac{1}{f_1^2} g(L, W) \left({}^2QK + \frac{(2-n_2)}{f_1} H_2^{f_1}(K) + 2(n_2-2) \frac{K(f_1)\nabla f_1}{f_1^2} \right. \right. \\
& \left. - n_1 \left(\frac{1}{f_2} H_2^{f_2}(K) - \frac{K(f_2)\nabla f_1 + K(f_1)\nabla f_2}{f_1 f_2} \right) - f_1^2 \Delta p \cdot K \right) \\
& - \frac{1}{f_1^2} g(K, W) \left({}^2QL + \frac{(2-n_2)}{f_1} H_2^{f_1}(L) + 2(n_2-2) \frac{L(f_1)\nabla f_1}{f_1^2} \right. \\
& \left. - n_1 \left(\frac{1}{f_2} H_2^{f_2}(L) - \frac{L(f_2)\nabla f_1 + L(f_1)\nabla f_2}{f_1 f_2} \right) - f_1^2 \Delta p \cdot L \right) \\
& + \left({}^2\text{Ric}(L, W) + h^p(L, W) + (1-n_2)h_2^p(L, W) + n_2L(p)W(p) \right. \\
& - g(L, W) \{ \Delta p + g(\nabla p, \nabla p) \} \\
& - n_1 \{ h_2^r(L, W) + L(r)W(r) - L(r)W(p) - L(p)W(r) \} \} K \\
& - \left({}^2\text{Ric}(K, W) + h^p(K, W) + (1-n_2)h_2^p(K, W) + n_2K(p)W(p) \right. \\
& - g(K, W) \{ \Delta p + g(\nabla p, \nabla p) \} \\
& \left. \left. - n_1 \{ h_2^r(K, W) + K(r)W(r) - K(r)W(p) - K(p)W(r) \} \right) \right\} L
\end{aligned}$$

from (35). Hence, for any $S \in \mathcal{L}(N_1)$, using (1) and (10) we get

$$g_2(H^p(K), S) = h_2^p(K, S) - 2K(p)S(p) + g(\nabla p, \nabla p)g_2(K, S),$$

and we obtain

$$\begin{aligned}
 g_2({}^2R_{KL}W, S) = & \{h_2^p(L, W) - W(p)L(p)\}g_2(K, S) - \{h_2^p(K, W) - W(p)K(p)\}g_2(L, S) \\
 & + \{g_2(H^p(K), S) + K(p)g_2(\nabla p, S)\}g(L, W) \\
 & - \{g_2(H^p(L), S) + L(p)g_2(\nabla p, S)\}g(K, W) \\
 & + \frac{1}{n-2} \left\{ \frac{1}{f_1^2} g(L, W) \left(g_2({}^2QK, S) + \frac{(2-n_2)}{f_1} g_2(H_2^{f_1}(K), S) \right. \right. \\
 & \left. \left. + 2(n_2-2) \frac{K(f_1)g_2(\nabla f_1, S)}{f_1^2} \right. \right. \\
 & \left. \left. - n_1 \left(\frac{1}{f_2} g_2(H_2^{f_2}(K), S) - \frac{K(f_2)g_2(\nabla f_1, S) + K(f_1)g_2(\nabla f_2, S)}{f_1 f_2} \right) \right. \right. \\
 & \left. \left. - f_1^2 \Delta p \cdot g_2(K, S) \right) \right. \\
 & - \frac{1}{f_1^2} g(K, W) \left(g_2({}^2QL, S) + \frac{(2-n_2)}{f_1} g_2(H_2^{f_1}(L), S) \right. \\
 & \left. \left. + 2(n_2-2) \frac{L(f_1)g_2(\nabla f_1, S)}{f_1^2} \right. \right. \\
 & \left. \left. - n_1 \left(\frac{1}{f_2} g_2(H_2^{f_2}(L), S) - \frac{L(f_2)g_2(\nabla f_1, S) + L(f_1)g_2(\nabla f_2, S)}{f_1 f_2} \right) \right. \right. \\
 & \left. \left. - f_1^2 \Delta p \cdot g_2(L, S) \right) \right. \\
 & + \left({}^2\text{Ric}(L, W) + h^p(L, W) + (1-n_2)h_2^p(L, W) + n_2 L(p)W(p) \right. \\
 & \left. - g(L, W)\{\Delta p + g(\nabla p, \nabla p)\} \right. \\
 & \left. - n_1 \{h_2^r(L, W) + L(r)W(r) - L(r)W(p) - L(p)W(r)\} \right) g_2(K, S) \\
 & - \left({}^2\text{Ric}(K, W) + h^p(K, W) + (1-n_2)h_2^p(K, W) + n_2 K(p)W(p) \right. \\
 & \left. - g(K, W)\{\Delta p + g(\nabla p, \nabla p)\} \right. \\
 & \left. - n_1 \{h_2^r(K, W) + K(r)W(r) - K(r)W(p) - K(p)W(r)\} \right) g_2(L, S) \Bigg\}.
 \end{aligned}$$

Hence we get

$$\begin{aligned}
 \frac{n_1}{n-2} {}^2\text{Ric}(L, W) + h_2^{\varphi_2}(L, W) = & \left\{ \frac{1}{n-2} \tau \right. \\
 & \left(f_1^4 n_2 + \frac{2(n_2-2)}{f_1^2(n-2)} - \frac{n_2^2 f_1^2}{n-2} \right) g(\nabla p, \nabla p) + \frac{2n_1}{(n-2)f_1^2} g(\nabla p, \nabla r) \\
 & + \left(f_1^2 + \frac{2-n_2}{n-2} \right) \Delta_2 p - \frac{n_1}{n-2} \Delta_2 r + \frac{2f_1^2(1-n_2)}{n-2} \Delta p \Bigg\} g_2(L, W) \\
 & + \left\{ 2 - n_2 - 2 \frac{n_2-2}{n-2} dp(L)dp(W) - \frac{n_1 f_2}{n-2 f_1^3} \left(dp(W)dr(L) + dp(L)dr(W) \right) \right\},
 \end{aligned}$$

if we contract the previous equation over K and S , where

$$\varphi_2 = - \left\{ \left(\frac{(3n_2 - n_2^2 - 1)}{n-2} + n_2 - f_1^2 - 1 \right) p + \frac{(2n_1 - 2)}{(n-2)} + \frac{n_1 - n_1 n_2}{n-2} r \right\}.$$

Thus, we obtain the assertion (b).

4. SOME APPLICATIONS OF WARPED-TWISTED PRODUCTS

Let (M, g_M) be a Riemannian manifold, and let f be a positive smooth function on J , where J is an open interval in $\mathbb{R}$. The product manifold $N = J \times_f M$ is called a *generalized Robertson–Walker spacetime* with dimension n , when it is given by the metric

$$g := -dt^2 \oplus f^2 g_M.$$

Here dt^2 is defined as the usual Euclidean metric tensor on J . For more details, see [Chen \(2014\)](#); [Mantica et al. \(2016\)](#); [Mantica et al. \(2016\)](#); [O’Neill \(1983\)](#).

Now, we generalize this notion. Let J be an open interval in $\mathbb{R}$, f_1 be a positive smooth function on J and f_2 be a positive smooth function on N_2 . The pseudo-Riemannian manifold ${}_2J \times_{f_1} N_2$ which is given by Lorentzian metric

$$g = -f_2^2 dt^2 \oplus f_1^2 g_2, \tag{36}$$

is called a *warped-twisted generalized Robertson-Walker spacetime* (see [Mantica and Molinari \(2021\)](#)). Here, ∂t is the standard vector field on J . Hence the following lemmas hold by (2)–(4) and Lemma 2.2 directly:

Lemma 4.1. *Let $f_2 J \times_{f_1} N_2$ be a warped-twisted generalized Robertson–Walker spacetime. Then we have*

$$\nabla_{\partial t} \partial t = f_2^2 \nabla r, \quad (37)$$

$$\nabla_L \partial t = \nabla_{\partial t} L = L(r) \partial t + p' L, \quad (38)$$

$$\nabla_K L = \nabla_K^2 L + K(p) L + L(p) K - g(K, L) \nabla p, \quad (39)$$

where $K, L \in \mathcal{L}(N_2)$, $p = \ln f_1$ and $r = \ln f_2$ are functions on $J \times N_2$, and also $p' = \partial t(p)$.

Lemma 4.2. *Let $f_2 J \times_{f_1} N_2$ be a warped-twisted generalized Robertson–Walker spacetime. Then we have*

$$R_{KL} \partial t = p' \{L(r) K - K(r) L\} + K(p') L - L(p') K, \quad (40)$$

$$R_{\partial t K} \partial t = \{-p'' + (p')^2\} K + p' K(r) \partial t - f_2^2 \{H^r(K) + K(r) \nabla r\}, \quad (41)$$

$$\begin{aligned} R_{K \partial t} L = & \{h_2^r(K, L) + K(r) L(r) - K(r) L(p) - K(p) L(r)\} \partial t \\ & + \{L(r) p' - \partial t(L(p))\} K + \{p' \nabla p + H^p(\partial t)\} g(K, L), \end{aligned} \quad (42)$$

where $K, L \in \mathcal{L}(N_2)$ and $h^p(\cdot, \cdot) = g(H^p(\cdot), \cdot)$.

Theorem 4.3. *Let $f_2 J \times_{f_1} N_2$ be a warped-twisted generalized Robertson–Walker spacetime given by the metric $g = -f_2^2 dt^2 \oplus f_1^2 g_2$. If it is conircularly flat, then for any $K, L \in \mathcal{L}(N_2)$*

(a) (N_2, g_2) is a gradient almost ζ -Yamabe soliton which holds $h_2^r = \lambda g_2 + \nu \zeta \otimes \zeta$, where $\zeta = dr$, $\nu = \frac{1}{(f_1 f_2)^2}$ and $\lambda =$

$$\frac{1}{f_2^2} \left\{ -p'' + (p')^2 - \frac{\tau f_2^2}{2n(n-1)} \right\},$$

(b) $p' \{L(r) K - K(r) L\} = \{K(p') L - L(p') K\}$.

Proof. Let $K, L \in \mathcal{L}(N_2)$. Using (27) and (41), we obtain

$$h_2^r(K, L) = \frac{1}{f_2^2} \left\{ -p'' + (p')^2 - \frac{\tau f_2^2}{2n(n-1)} \right\} g_2(K, L) + \frac{1}{(f_1 f_2)^2} dr(K) dr(L).$$

Thus, the assertion (a) holds. On the other hand, the assertion (b) follows directly from (27) and (40).

Let J be an open interval in $\mathbb{R}$ and dt^2 be the Euclidean metric tensor on J . Let (B, g_B) be a Riemannian manifold, and let $f : B \rightarrow (0, \infty)$ be a smooth function. The product manifold $N = J \times B$ is called a *standard static spacetime* when it is given by the metric tensor

$$g := -f^2 dt^2 \oplus g_B.$$

Then, it is denoted by $f J \times B$. For more details see [Allison and Ünal \(2003\)](#); [Beem et al. \(1996\)](#); [Dobarro and Ünal \(2004\)](#).

Now we generalize this notion. Let J be a real open interval, and let f_1 be a positive smooth function on N . Suppose that f_2 depends only on the points of J . Consider a pseudo-Riemannian manifold $N = J \times N_2$, which is given by the Lorentzian metric

$$g := -f_1^2 dt^2 \oplus f_2^2 g_2, \quad (43)$$

with ∂t denoting the standard vector field on J . Then we say that $f_1 J \times_{f_2} N_2 = (N, g)$ is a *warped-twisted standard static spacetime*. Hence we give:

Lemma 4.4. *Let $f_1 J \times_{f_2} N_2$ be a warped-twisted standard static spacetime. Then we have*

$$\nabla_{\partial t} \partial t = 2p' \partial t + f_1^2 \nabla p, \quad (44)$$

$$\nabla_L \partial t = \nabla_{\partial t} L = L(p) \partial t + r' L, \quad (45)$$

$$\nabla_K L = \nabla_K^2 L - g(K, L) \nabla r, \quad (46)$$

where $K, L \in \mathcal{L}(N_2)$, $p = \ln f_1$ and $r = \ln f_2$ are functions on $J \times N_2$, and also $\partial t(p) = p'$ and $\partial t(r) = r'$.

Lemma 4.5. Let $_{f_1} J \times_{f_2} N_2$ be a warped-twisted standard static spacetime. Then we have

$$R_{KL} \partial t = r' \{L(p)K - K(p)L\}, \quad (47)$$

$$R_{\partial t K} \partial t = \{-r'' + (r')^2 - 2r'p'\}K + \{r'K(p) - K(p')\}\partial t - f_1^2 \{K(p)\nabla p + H^p(K)\}, \quad (48)$$

$$R_{K\partial t} L = \{h_2^p(K, L) + K(p)L(p)\}\partial t + L(p)r'K + \{r'\nabla r + H^r(\partial t)\}g(K, L), \quad (49)$$

$$R_{KL}W = {}^2R_{KL}W + H^r(L)g(K, W) - H^r(K)g(L, W), \quad (50)$$

where $K, L, W \in \mathcal{L}(N_2)$.

Theorem 4.6. Let $_{f_1} J \times_{f_2} N_2$ be a warped-twisted standard static spacetime given by the metric $g = -f_1^2 dt^2 \oplus f_2^2 g_2$. If it is concircularly flat, then for any $K, L \in \mathcal{L}(N_2)$

(a) (N_2, g_2) is an Einstein manifold with ${}^2\text{Ric} = \lambda g_2$, where $\lambda = \frac{\tau f_2^2}{2n(n-1)}(n_2 - 1)$,

(b) (N_2, g_2) is a gradient almost ζ -Yamabe soliton with $h_2^p(K, L) = \lambda_2 g_2(K, L) + \nu \tilde{d}p \otimes \tilde{d}p$, where $\lambda_2 = \frac{1}{f_1^2} \left\{ \frac{\tau f_1^2}{2n(n-1)} - r'' + (r')^2 - 2r'p' \right\}$, $\nu = -\frac{1}{(f_1 f_2)^2}$ and $\zeta = \tilde{d}p$, where $\tilde{d}p$ is the pullback of dp to N_2 ,

(c) $L(p)K - K(p)L = 0$ or $_{f_1} J \times N_2$ is a twisted product.

Proof. Let $K, L \in \mathcal{L}(N_2)$. Using (27) and (50), we have

$${}^2\text{Ric}(K, L) = \left\{ \frac{\tau f_2^2}{2n(n-1)}(n_2 - 1) \right\} g_2(K, L),$$

which proves the assertion (a). On the other hand, using (27) and (48), we obtain

$$h_2^p(K, L) = -\frac{1}{f_1^2} \left\{ \frac{\tau f_1^2}{2n(n-1)} - r'' + (r')^2 - 2r'p' \right\} g_2(K, L) - \frac{1}{(f_1 f_2)^2} \tilde{d}p(K) \tilde{d}p(L),$$

which yields (b). Also, using (27) and (47), we find $r' \{L(p)K - K(p)L\} = 0$. Hence, $L(p)K - K(p)L = 0$ or $r' = 0$. If $r' = 0$, then $r = \ln f_2$ is a constant. If $f_2 = c$, then $g = -f_1^2 dt^2 \oplus c^2 g_2$, which says that the manifold is a twisted product of type $_{f_1} J \times N_2$. Thus, the assertion (c) holds.

Peer Review: Externally peer-reviewed.

Conflict of Interest: Author declared no conflict of interest.

Financial Disclosure: This work is supported by 1001-Scientific and Technological Research Projects Funding Program of The Scientific and Technological Research Council of Türkiye (TUBITAK) project number 119F179.

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