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Construction of complete $(k,3)$ -arcs containing Menelaus 6-figures in $PG(2,4)$ using a secant-based algorithm

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Abstract

This study investigates complete $(k,3)$ -arcs containing a Menelaus 6-figure in the projective plane $PG(2,4)$. We developed and implemented a computational algorithm in C# to construct and classify complete arcs through a detailed analysis of the secant line distributions associated with the Menelaus 6-figure. The algorithm identifies seven points outside the Menelaus 6-figure, which are classified into two categories based on their incidence with 0-secant, 1-secant, and 2-secant lines. By analyzing the extension of this configuration, one unique complete $(7,3)$ -arc and eight distinct complete $(9,3)$ -arcs were identified, all of which contain the Menelaus 6-figure.

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Keywords: projective plane; $(k,3)$ -arc; Menelaus 6-figure; finite geometry; $PG(2,4)$

1. Introduction

A projective plane P is characterized by a collection N of points and a collection D of subsets of N , which denote lines. Four points can define six separate lines in this arrangement, and every pair of points lies on exactly one line. Additionally, every pair of distinct lines connects at exactly one point. A subset B of points and lines that also constitutes a projective plane is regarded as a subplane of the projective plane P when we analyze the incidence relation within P .

Arcs are extensively utilized in combinatorics and related disciplines and are essential to projective geometry. In a finite projective plane π , a k -arc is a set K of k ($k \geq 3$) points such that no three points in K are collinear. If the plane π

has order p , then $k \geq p+2$, but the maximum value of k can only be reached if p is even. A $(p+2)$ -arc is referred to as a hyperoval. Ovals are commonly discussed in the literature, and Hirschfeld is one source that does so [1].

The literature on arcs in projective planes is rich, particularly concerning complete $(k,2)$ -arcs, often involving complete quadrangles that generate the Fano plane. These configurations have been thoroughly examined in various studies [2, 3]. The investigation of $(k,3)$ -arcs related to complete quadrangles in $PG(2,5)$ has also been presented in [4], while arc-finding algorithms implemented in C# have been applied to the classification of $(k,2)$ -arcs, associated with triangles and quadrangles in $PG(2,4)$ over $GF(4)$ [5].

In parallel, the classical theorems of Ceva and Menelaus have played a central role in the geometric foundations of projective configurations. Ceva's Theorem characterizes the concurrency of Cevians, while Menelaus's Theorem provides conditions for point collinearity—both remaining valid under projective transformations through directed and cross ratios. These theorems are foundational in the study of harmonic sets, perspectivity, and duality, and have broad implications in modern projective geometry.

The Benyi–Curgus generalization of Ceva's and Menelaus's theorems simultaneously characterizes collinearity and concurrency based on six points on a triangle's sides [6]. Benitez [7] further addressed the affine-projective disconnect by reformulating Ceva's Theorem using the projective invariant cross ratio, while Menelaus's Theorem was reframed via its dual. Nicolae [8] investigated the Ceva–Menelaus transformation, showing its image as conics tangent to triangle sides, and demonstrated that the harmonic transform of a cevian forms a pencil of lines.

Further generalizations of these classical results were made by Funk [9], who formulated them in projective planes $P_2(F)$, over fields of characteristic not equal to two. Kaya and Çiftçi [10] introduced Ceva and Menelaus 6-figures in Moufang planes, which were later extended to fibered geometries involving graded incidence structures [11]. Building upon this, Akça et al. [12] developed intuitionistic fuzzy projective analogues of the classical conditions, expanding their applicability to more abstract settings.

Building on these foundational results, recent studies have focused on the construction of complete $(k,3)$ -arcs in $PG(2,5)$ containing a Menelaus configuration, employing algorithmic approaches based on secant incidence classifications [13,14]. One study investigates complete $(11,3)$ -arcs, identifying thirteen remaining points categorized by their 0-, 1-, and 2-secant interactions, and deriving eighteen distinct arcs through two structured construction methods [13]. Another explores complete $(10,3)$ -arcs using a C#-based algorithm, generating thirty-two arcs by similarly analyzing secant distributions [14]. While previous research has concentrated mainly on $PG(2,5)$ or on general $(k,2)$ -arcs, our work extends this direction by focusing on $PG(2,4)$ and systematically constructing complete $(k,3)$ -arcs that contain a Menelaus 6-figure. This integration of a Menelaus configuration as a combinatorial constraint allows the identification of new classes of arcs—most notably, one unique complete $(7,3)$ -arc and eight distinct complete $(9,3)$ -arcs—which complement and expand the known classification of complete arcs in small-order projective planes. By doing so, the work fills a gap in the classification of arcs for small-order planes and illustrates how algorithmic methods can be effectively adapted to special incidence configurations.

The main objective of this study is to construct and classify complete $(k,3)$ -arcs which contain the points of a Menelaus 6-figure in the projective plane $PG(2,4)$. By applying the computational algorithm developed in C#, the study identifies the points outside the Menelaus 6-figure. Leveraging this algorithm, the construction methods are introduced, which lead to the identification of incomplete $(6,3)$ -arcs containing the points of Menelaus 6-figure, only one complete $(7,3)$ -arc containing a given Menelaus 6-figure, and eight complete $(9,3)$ -arcs; containing all points of Menelaus 6-figure and three remaining points.

2. Preliminaries

Key concepts in projective geometry are covered in this part, such as projective planes, finite fields, arc properties, and a Menelaus 6-figure. The basis for our investigation is provided by these definitions.

Definition 1. In a projective plane (N, D, ϕ) , the set N represents points and D denotes lines (subsets of N), with the following axioms holding: every pair of points lies on one and only one line, each pair of distinct lines meets at exactly one point, and there are four non-collinear points, no three of which lie on a common line.

Definition 2. [1] The vector space $V(n+1, q)$ is $(n+1)$ -dimensional and consists of vectors with coordinates from the finite field $GF(q)$. The projective space $PG(n, q)$ is defined as the collection of points, each corresponding to a line that passes through the origin in $V(n+1, q)$. Specifically, this means that each point in $PG(n, q)$ can be represented as $P(x)$, where x is any non-zero vector in $V(n+1, q)$. If K is the finite field $GF(q)$, also denoted as F_q , then the n -dimensional projective plane is referred to as $PG(n, K)$ or $PG(n, q)$. In this context, q represents the order of $PG(n, q)$. The number of points in this projective plane can be determined using the formula

$$\theta(n) = \frac{q^{n+1}-1}{q-1} \quad (1)$$

$(x_1, x_2, \dots, x_n)$ represents a point in N , where $x_1, x_2, \dots, x_n$ are not all zero, and $(\lambda x_1, \lambda x_2, \dots, \lambda x_n) \equiv (x_1, x_2, \dots, x_n)$, $\lambda \in K \setminus \{0\}$. Similarly, the notation $[a_1, a_2, \dots, a_n]$ denotes any line in D , where $a_1, a_2, \dots, a_n$ are not all zero. The relationship $[\mu a_1, \dots, \mu a_n] \equiv [a_1, \dots, a_n]$ holds for $\mu \in K \setminus \{0\}$. The projective plane P_2K is characterized as a point-line geometry $(N, D, \circ)$ defined by K . The incidence relation is given by $\circ$:

$$(x_1, \dots, x_n) \circ [a_1, \dots, a_n] \Leftrightarrow a_1 x_1 + a_2 x_2 + a_3 x_3 + \dots + a_n x_n = 0 \quad (2)$$

Let p denote a prime number and r a positive integer. The projective plane of order $q = p^r$ over the finite field $K = GF(p^r)$, where p^r represents the number of elements, is expressed as $P_2K = PG(2, p^r)$.

Theorem 1. Let $\mathcal{P}$ be a projective plane. A 6-figure in $\mathcal{P}$ is a sequence of six distinct points $(A_1 A_2 A_3, B_1 B_2 B_3)$ such that $A_1 A_2 A_3$ constitutes a nondegenerate triangle with $B_1 \in \langle A_2, A_3 \rangle$, $B_2 \in \langle A_1, A_3 \rangle$, and $B_3 \in \langle A_1, A_2 \rangle$. The points $A_1, A_2, A_3, B_1, B_2, B_3$ are called vertices of this 6-figure. Such a configuration is said to be a Menelaus 6-figure if B_1, B_2 , and B_3 are collinear [11].

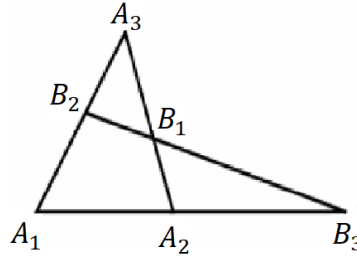


Fig. 1. Menelaus 6-figure

Definition 4. [15] In a projective plane, a (k, n) –arc $\mathcal{K}$ is a set of k points such that some line intersects $\mathcal{K}$ in n points, and no line intersects $\mathcal{K}$ in more than n points, where $n \geq 2$.

Definition 5. [16] A line l of projective plane is an μ –secant of (k, n) –arc $\mathcal{K}$ if l intersects $\mathcal{K}$ in μ points. Let τ_i be the total number of i –secants to $\mathcal{K}$. The number of i –secants to $\mathcal{K}$ through a point Q of $\mathcal{P} \setminus \mathcal{K}$ is denoted by σ_i or $\sigma_i(Q)$. Moreover, a point Q of $\mathcal{P} \setminus \mathcal{K}$ is called point of index zero if $\sigma_n(Q) = 0$.

If there isn't a $(k+1, n)$ –arc that contains a (k, n) –arc, then it is complete.

Definition 6. Let K be a (k, n) –arc in a projective plane. A point not contained in K but lying on an i –secant of K —that is, a line intersecting K in exactly i distinct points—is called a point of index i [16].

3. Construction of $PG(2,4)$

The irreducible polynomial $f(x) = x^2 + x + 1$, where the elements $0, 1, t, t^2$ of $GF(4)$, is used to construct $PG(2,4)$, which has 21 points and 21 lines over $GF(4)$. In $PG(2,4)$, each line has five points, and each point has five lines that pass through it [17].

The projective plane $PG(2,4)$ has a point set N that is $N = \{N_i | i = 1, 2, \dots, 21\}$, where $N_1 = (0,1,0), N_2 = (0,0,1), N_3 = (0,1,1), N_4 = (0,1,t^2), N_5 = (0,1,t), N_6 = (1,1,1), N_7 = (1,0,1), N_8 = (1,t,1), N_9 = (1,t^2,1), N_{10} = (1,1,0), N_{11} = (1,1,t^2), N_{12} = (1,1,t), N_{13} = (1,t,t^2), N_{14} = (1,t^2,t), N_{15} = (1,0,0), N_{16} = (1,t,0), N_{17} = (1,t^2,0), N_{18} = (1,0,t^2), N_{19} = (1,0,t), N_{20} = (1,t^2,t^2), N_{21} = (1,t,t)$.

Table 1 provides the $PG(2,4)$ incident relation table. Each row represents points located on the line $D_i, i = 1, 2, \dots, 21$.

Table 1. The lines of $PG(2,4)$

D_1	N_1	N_2	N_3	N_4	N_5
D_2	N_1	N_6	N_7	N_8	N_9
D_3	N_1	N_{10}	N_{15}	N_{16}	N_{17}
D_4	N_1	N_{11}	N_{13}	N_{18}	N_{20}
D_5	N_1	N_{12}	N_{14}	N_{19}	N_{21}
D_6	N_2	N_6	N_{10}	N_{11}	N_{12}
D_7	N_2	N_7	N_{15}	N_{18}	N_{19}
D_8	N_2	N_9	N_{14}	N_{17}	N_{20}
D_9	N_2	N_8	N_{13}	N_{16}	N_{21}
D_{10}	N_3	N_6	N_{15}	N_{20}	N_{21}
D_{11}	N_5	N_6	N_{14}	N_{16}	N_{18}
D_{12}	N_4	N_6	N_{13}	N_{17}	N_{19}
D_{13}	N_3	N_7	N_{10}	N_{13}	N_{14}
D_{14}	N_3	N_8	N_{12}	N_{17}	N_{18}
D_{15}	N_3	N_9	N_{11}	N_{16}	N_{19}
D_{16}	N_4	N_7	N_{12}	N_{16}	N_{20}
D_{17}	N_5	N_7	N_{11}	N_{17}	N_{21}
D_{18}	N_5	N_8	N_{10}	N_{19}	N_{20}
D_{19}	N_4	N_9	N_{10}	N_{18}	N_{21}
D_{20}	N_5	N_9	N_{12}	N_{13}	N_{15}
D_{21}	N_4	N_8	N_{11}	N_{14}	N_{15}

4. Investigation of complete $(k,3)$ -Arcs based on the Menelaus 6-figure in $PG(2,4)$

This section presents an algorithm used to construct complete $(k,3)$ -arcs by including all points of a Menelaus 6-figure in $PG(2,4)$.

4.1. Methodology: Constructing $(k,3)$ -arcs in $PG(2,4)$

This section presents an algorithm designed to construct $(k,3)$ -arcs, incorporating the Menelaus 6-figure in $PG(2,4)$. The algorithm follows a structured approach, ensuring systematic identification and extension of arcs.

Before starting the methodological approach, the construction of the projective plane and the choice of the Menelaus 6-figure, which are necessary for all methodologies, must be performed as follows:

Step 1. Initialization and Projective Plane Definition

- With the help of an irreducible polynomial $f(x) = x^2 + x + 1$ over $GF(4)$, 21 points and 21 lines of $PG(2,4)$ are determined (see Table 1).
- The identified points and lines are stored in dataset A.

Step 2. Definition of the Menelaus 6-figure

- A Menelaus 6-figure M is selected as consisting of six points $A_1, A_2, A_3, B_1, B_2, B_3$ in $PG(2,4)$.
- When the algorithm is applied to M , it is seen that M forms an incomplete $(6,3)$ -arc in $PG(2,4)$.

Step 3. Determination of Remaining Points

- When the algorithm is applied by taking the all points of selected M in $PG(2,4)$, there are seven points that can be added back.
- These seven points are classified into two categories based on their secant line properties to Menalaus 6-figure M in $PG(2,4)$:
 - There is one point in Category 1 such that $\sigma_0(C) = 2, \sigma_2(C) = 3$.
 - There are six points in Category 2 such that $\sigma_2(D_k) = 1, \sigma_1(D_k) = 4, k = 1, 2, \dots, 6$

Step 4: Constructing complete $(7,3)$ -Arcs using the point from Category 1

By adding the point C in Category 1 such that $\sigma_0(C) = 2, \sigma_2(C) = 3$ to the incomplete $(6,3)$ -arc be constructed the points of Menelaus 6-figure, the complete $(7,3)$ -arc is obtained as $M \cup \{C\}$.

Step 5: Constructing complete $(9,3)$ -Arcs using the points from Category 2

By selecting three points from $D_k, k = 1, 2, \dots, 6$ in Category 2 — each lying on a distinct 2-secant line — and adding them to the incomplete $(6,3)$ -arc defined by Menelaus 6-figure, a complete $(9,3)$ -arc is obtained.

4.2. Algorithm implementation

We present the following algorithm, implemented in $C\#$, designed to identify complete $(k,3)$ -arcs within $PG(2,4)$:

```

A ← Read(Excel File)
B ← Read(Text File)
C ← A
while s(C) > 0
    Bi ← input(b), {b | b ∈ C, b ∉ B, i = s(B) + 1}
    j = 1
    while j ≤ s(B)
        for k = (j+1) to s(B)
            m ← the index of row on Bj, Bk
            D ← Amn; {Amn | Amn ≠ Bj, Amn ≠ Bk, n = 1, ..., 10}
            Remove a from A; {a | a ∈ A, a ∈ D}
            C ← C; {c | c ∈ A, c ∉ C}
    
```

```

end for
j=j+1
end while
end while

```

Theorem 2. A Menelaus 6-figure determines an incomplete $(6,3)$ -arc in $PG(2,4)$.

Proof. Let $M = (A_1A_2A_3, B_1B_2B_3)$ be any Menelaus 6-figure in $PG(2,4)$. M contains six lines and six points. From Theorem 1, since no four points in the Menelaus 6-figure are collinear, the set of these points forms a $(6,3)$ -arc. From Definition 6, this $(6,3)$ -arc is not complete.

Now, if the given algorithm is applied to the points of the incomplete $(6,3)$ -arc determined by Menelaus 6-figure to find $(k,3)$ -arcs, then there are seven remaining points.

Theorem 3. There is only one complete $(7,3)$ -arc containing a given Menelaus 6-figure in $PG(2,4)$.

Proof. Let $M = (A_1A_2A_3, B_1B_2B_3)$ be any Menelaus 6-figure in $PG(2,4)$. M contains six lines and six points. If the algorithm is applied to M , seven points remain. Six of these points have four 1-secants and one 2-secant. Only one of these points has two 0-secants and three 2-secants. If this point with the maximum 2-secant among these remaining points is added to the incomplete $(6,3)$ -arc constructed by M , the $(7,3)$ -arc is obtained. When the algorithm is applied to the points of the obtained $(7,3)$ -arc, no further points can be added to it. Therefore, the resulting $(7,3)$ -arc is complete. Since the arcs generated by adding the other remaining points using the algorithm are incomplete $(7,3)$ -arcs, the complete $(7,3)$ -arc containing a specific Menelaus 6-figure is unique.

Theorem 4. There are eight different complete $(9,3)$ -arcs containing a given Menelaus 6-figure in $PG(2,4)$.

Proof. Let $M = (A_1A_2A_3, B_1B_2B_3)$ be any Menelaus 6-figure in $PG(2,4)$. According to Theorem 2, the Menelaus 6-figure constitutes an incomplete $(6,3)$ -arc. When the algorithm is applied to M , seven candidate points remain that can potentially be added to the incomplete $(6,3)$ -arc. By Theorem 3, if the point with the maximum 2-secant number is added to M , a complete $(7,3)$ -arc is obtained. The other remaining points to be added to incomplete $(6,3)$ -arc are denoted by D_k , where $k = 1, 2, \dots, 6$ and each satisfies the conditions $\sigma_2(D_k) = 1, \sigma_1(D_k) = 4$. These six points lie on three distinct lines, each intersecting the Menelaus 6-figure in exactly two points—that is, they are 2-secant lines, each containing a pair of the D_k points. If one point from each of these three 2-secant lines is added to the incomplete $(6,3)$ -arc, a $(9,3)$ -arc is obtained. Upon applying the algorithm to these nine points, no further points can be added, and thus the $(9,3)$ -arc is complete. Considering the combinatorial choices—two possibilities per line—there are, $2 \cdot 2 \cdot 2 = 8$ distinct ways to select one point from each of the three lines. Therefore, eight different complete $(9,3)$ -arcs are obtained through this construction.

Corollary. An incomplete $(6,3)$ -arc determined by a Menelaus 6-figure in $PG(2,4)$ can be extended either to the unique complete $(7,3)$ -arc by adding the point with the maximal 2-secant number, or to eight distinct complete $(9,3)$ -arcs by adding one point from each of the three 2-secant lines intersecting the Menelaus 6-figure, other than the point with the maximal 2-secant number.

Example 1.

Let M be a Menelaus 6-figure $(N_2N_6N_{20}, N_{15}N_{17}N_{10})$. If the points of the Menelaus 6-figure are taken in $PG(2,4)$, and the algorithm is applied, then $N_4, N_5, N_7, N_8, N_{13}, N_{18}, N_{19}$ points are remained.

According to our methodologies, by adding the point N_{19} in Category 1 such that $\sigma_0(N_{19}) = 2, \sigma_2(N_{19}) = 3$ to the incomplete $(6,3)$ -arc be constructed the points of Menelaus 6-figure, and applying the algorithm algorithm is applied to all points of M and N_{19} , it is seen that there isn't any remaining point and $M_1 = M \cup \{N_{19}\}$ is a complete $(7,3)$ -arc in $PG(2,4)$.

Example 2.

Let M be a Menelaus 6-figure $(N_2N_6N_{20}, N_{15}N_{17}N_{10})$. If the points of the Menelaus 6-figure are taken in $PG(2,4)$, and the algorithm is applied, then $N_4, N_5, N_7, N_8, N_{13}, N_{18}, N_{19}$ points are remained.

According to our methodologies, by successively adding N_4, N_5 , and finally N_7 in Category 2 such that the points of Category 2 have four 1-secant, one 2-secant to the incomplete $(6,3)$ -arc be constructed the points of Menelaus 6-figure, and applying the algorithm at each step, the incomplete arcs grow as follows:

$$M_2 = \{N_2, N_4, N_6, N_{10}, N_{15}, N_{17}, N_{20}\} - a(7,3) - arc$$

$$M_3 = \{N_2, N_4, N_5, N_6, N_{10}, N_{15}, N_{17}, N_{20}\} - a(8,3) - arc$$

Finally, the addition of N_7 results in the complete $(9,3)$ -arc:

$$M_4 = \{N_2, N_4, N_5, N_6, N_7, N_{10}, N_{15}, N_{17}, N_{20}\}.$$

5. Conclusion

In this study, the structural properties of complete $(k,3)$ -arcs formed in the projective plane $PG(2,4)$, which include Menelaus 6-figure and are classified the secant lines are systematically investigated by developing and executing a computational algorithm in C#.

The six points forming a Menelaus 6-figure define an incomplete $(6,3)$ -arc in the finite projective plane $PG(2,4)$.

Moreover, several distinct types of complete $(k,3)$ -arcs were identified, depending on the selection of 2-secant lines. Specifically:

- One complete $(7,3)$ -arcs includes all points of a given Menelaus 6-figure along with one additional point.
- Eight distinct complete $(9,3)$ -arcs include all points of a given Menelaus 6-figure along with three additional points.

In accordance with the combinatorial structure of $PG(2,4)$, no complete $(8,3)$ -arc including the points of a Menelaus 6-figure exists, as confirmed by our computational results.

Most studies in the literature provide general classifications of complete arcs, and this work specifically focuses on those containing a Menelaus 6-figure. Imposing this additional combinatorial constraint leads to a precisely characterized and well-defined set of solutions, consisting of a unique $(7,3)$ -arc and exactly eight $(9,3)$ -arcs that had not been explicitly enumerated previously.

The algorithm presented here, though applied to $PG(2,4)$, is general in nature. With suitable modifications to the initial incidence data, it can be extended to other finite projective planes $PG(2,q)$ with $q > 4$, enabling the systematic study of complete $(k,3)$ -arcs under various combinatorial constraints. Moreover, the method can be adapted to investigate arcs containing other special configurations, such as quadrangles, and generalized to explore (k,n) -arcs in higher-dimensional projective spaces $PG(n,q)$. These potential extensions highlight the flexibility of the proposed approach and its value as a framework for future research in finite geometry.

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