

TWO FIXED POINT RESULTS ON b -METRIC SPACE

ISHAK ALTUN*, AND DILDAR GÜLSEMIH AKEKIN**

*KIRIKKALE UNIVERSITY, TÜRKİYE. ORCID: 0000-0002-7967-0554

**KIRIKKALE UNIVERSITY, TÜRKİYE. ORCID: 0000-0002-8878-4086.

ABSTRACT. In this paper, we present two fixed point theorems on b -metric spaces. First, we obtain a general result, which includes many fixed point theorems on b -metric space in the literature as a special case, by using implicit relation technique. Then, we define the concept of nearly Lipschitzian mapping in b -metric spaces, and we obtain a fixed point theorem for such mappings.

1. INTRODUCTION AND PRELIMINARIES

The notion of metric, which has an important place in mathematics, has been generalized in several ways by weakening some of the metric axioms [2, 3, 10]. For example, by weakening the axiom that the necessary and sufficient condition for two points to be equal is that the distance between them is zero, the concepts of pseudometric have been introduced. Again by neglecting the symmetry axiom and triangular inequality axiom respectively the notion of quasimetric and the notion of semimetric have been defined. On the other hand, by weakening the triangle inequality, the concept of b -metric has been introduced, which is a concept that has been frequently studied recently.

Czerwik in [6] introduced the concept of b -metric space (in short b -ms).

Definition 1.1 ([6]). *Let B be a non-empty set and $s \geq 1$ be a fixed real number. A function $b : B \times B \rightarrow \mathbb{R}_+ = [0, \infty)$ is called a b -metric (in short b -m) on B if, for all $\omega, \varpi, v \in B$, the following conditions hold:*

- (b1) $b(\omega, \varpi) = 0$ if and only if $\omega = \varpi$,
- (b2) $b(\omega, \varpi) = b(\varpi, \omega)$,
- (b3) $b(\omega, v) \leq s [b(\omega, \varpi) + b(\varpi, v)]$.

We call the triplet (B, b, s) a b -ms.

It is important to note that the collection of b -ms's is strictly broader than that of metric spaces, as a b -metric reduces to a metric when $s = 1$.

There are many examples of b -ms in the literature. Let us present one example here for the sake of completeness.

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Example 1.1 ([1]). Let (B, ρ) be a metric space, $p > 1$ be a real number and let $b(\omega, \varpi) = (\rho(\omega, \varpi))^p$. Then the triplet $(B, b, 2^{p-1})$ is a b -ms. Indeed (b1) and (b2) are obviously hold. On the other hand, since the function $f(\omega) = \omega^p$ ($\omega \geq 0$) is convex, then

$$\left(\frac{\lambda + \mu}{2}\right)^p \leq \frac{1}{2}(\lambda^p + \mu^p),$$

and hence

$$(\lambda + \mu)^p \leq 2^{p-1}(\lambda^p + \mu^p)$$

holds for $\lambda, \mu \geq 0$. Thus, for each $\omega, \varpi, v \in B$ we obtain

$$b(\omega, \varpi) = (\rho(\omega, \varpi))^p \leq 2^{p-1}[b(\omega, v) + b(v, \varpi)].$$

So the condition (b3) of Definition 1.1 is hold and so b is a b -metric.

Definition 1.2 ([4]). Let (B, b, s) be a b -ms. A sequence $\{\omega_n\}$ in B is defined as:

- (a) convergent if there exists an element $\omega \in B$ such that $b(\omega_n, \omega) \rightarrow 0$ as $n \rightarrow \infty$. In this case, we write $\lim_{n \rightarrow \infty} \omega_n = \omega$.
- (b) Cauchy if $b(\omega_n, \omega_m) \rightarrow 0$ as $n, m \rightarrow \infty$, meaning that for every $\varepsilon > 0$, there is a natural number n_0 such that for all $n, m \geq n_0$, $b(\omega_n, \omega_m) < \varepsilon$.

The b -ms (B, b, s) is considered complete (in short cb -ms) if every Cauchy sequence in B converges.

Proposition 1.1 ([4], Remark 2.1). Consider a b -ms (B, b, s) . The following statements are true:

- (i) A convergent sequence has at most one limit,
- (ii) Every convergent sequence is also Cauchy,
- (iii) In general, a b -metric does not exhibit continuity.

Although the b -metric function is not continuous in general, it satisfies the following property.

Lemma 1.2 ([1]). Let (B, b, s) be a b -ms, and assume that the sequences $\{\omega_n\}$ and $\{\varpi_n\}$ converge to ω and ϖ , respectively. Then, the following inequality holds:

$$\frac{1}{s^2}b(\omega, \varpi) \leq \liminf_{n \rightarrow \infty} b(\omega_n, \varpi_n) \leq \limsup_{n \rightarrow \infty} b(\omega_n, \varpi_n) \leq s^2b(\omega, \varpi).$$

In particular, if $\omega = \varpi$, we have $\lim_{n \rightarrow \infty} b(\omega_n, \varpi_n) = 0$.

Furthermore, for any $u \in B$, the following inequality holds:

$$\frac{1}{s}b(\omega, u) \leq \liminf_{n \rightarrow \infty} b(\omega_n, u) \leq \limsup_{n \rightarrow \infty} b(\omega_n, u) \leq sb(\omega, u).$$

The aim of this paper is to present some fixed point theorems in b -metric space inspired by metric fixed point theory. As it is known, there are basically three stages in the proof of metric fixed point theorems. The first is to construct an iteration sequence, which is usually a Picard iteration sequence, the second stage is to show that this sequence is a Cauchy sequence, which is the real difficulty in the proof. The third is to show that the point to which the sequence converges, which is guaranteed by the completeness of the space, is a fixed point. In showing that the sequence in question is a Cauchy sequence, sometimes direct proof, sometimes proof by contradiction method is used according to the hypotheses. For example, in a metric space (B, ρ) if there exists $\lambda \in [0, 1)$ such that

$$\rho(\omega_n, \omega_{n+1}) \leq \lambda \rho(\omega_{n-1}, \omega_n) \tag{1.1}$$

for all $n \in \mathbb{N}$, then $\{\omega_n\}$ is a Cauchy sequence. Indeed, by using (1.1) and the triangular inequality of the metric ρ , we have

$$\begin{aligned}\rho(\omega_n, \omega_m) &\leq \rho(\omega_n, \omega_{n+1}) + \rho(\omega_{n+1}, \omega_{n+2}) + \cdots + \rho(\omega_{m-1}, \omega_m) \\ &\leq \frac{\lambda^n}{1-\lambda} \rho(\omega_0, \omega_1),\end{aligned}$$

for $m > n$, which shows that $\{\omega_n\}$ is Cauchy. Similarly, by using the triangular inequality of the metric ρ , we have, for $m > n$

$$\begin{aligned}\rho(\omega_n, \omega_m) &\leq \rho(\omega_n, \omega_{n+1}) + \rho(\omega_{n+1}, \omega_{n+2}) + \cdots + \rho(\omega_{m-1}, \omega_m) \\ &\leq \sum_{k=n}^{\infty} \rho(\omega_k, \omega_{k+1})\end{aligned}$$

and so the convergence of the series

$$\sum_{n=1}^{\infty} \rho(\omega_n, \omega_{n+1}) \quad (1.2)$$

is guaranteed that $\{\omega_n\}$ is a Cauchy sequence. Therefore, before presenting a fixed point theorem in a b -ms, it is important to investigate whether these situations are also valid in a b -ms.

As a result of the research carried out in this context, Suzuki [12] presented an example showing that the convergence of the series given in (1.2) in a b -ms is not sufficient for the sequence $\{\omega_n\}$ to be a Cauchy. However, the following lemmas helps us in this direction.

Lemma 1.3 ([7]). *Let $\{\omega_n\}$ be a sequence in a b -ms (B, b, s) . The sequence $\{\omega_n\}$ is Cauchy if there exists a constant $\gamma > \log_2 s$ such that the series*

$$\sum_{n=1}^{\infty} n^\gamma b(\omega_n, \omega_{n+1})$$

is convergent.

Lemma 1.4 ([7]). *Let $\{\omega_n\}$ be a sequence in a b -ms (B, b, s) . The sequence $\{\omega_n\}$ is Cauchy if there exists $\alpha > 1$ such that the series*

$$\sum_{n=1}^{\infty} \alpha^n b(\omega_n, \omega_{n+1})$$

is convergent.

On the other hand, it may be thought that the inequality (1.1) may not be sufficient for the sequence $\{\omega_n\}$ to be a Cauchy sequence in a b -metric space, and in fact it is clear that $\lambda < \frac{1}{s}$ will guarantee this. However, the following lemma shows that this is valid even if $\lambda < 1$.

Lemma 1.5 ([8, 12]). *Consider (B, b, s) , a b -ms, and a sequence $\{\omega_n\}$ within B . Suppose there exists $\alpha \in [0, 1)$ such that*

$$b(\omega_n, \omega_{n+1}) \leq \alpha b(\omega_{n-1}, \omega_n),$$

for any $n \in \mathbb{N}$. Under this condition, the sequence $\{\omega_n\}$ is Cauchy.

Considering the above lemmas and similar ones, many fixed point theorems have been obtained in b -ms. Let us mention a few here.

Throughout the remainder of this paper, the term "fixed point" will be abbreviated as "fp" for brevity.

Theorem 1.6 ([9]). *Let (B, b, s) be a cb-ms, and let $\Upsilon : B \rightarrow B$ be a mapping such that:*

$$b(\Upsilon\omega, \Upsilon\varpi) \leq \lambda b(\omega, \varpi)$$

for all $\omega, \varpi \in B$, where $\lambda \in [0, 1)$. Then Υ admits a unique fp v , and for any starting point $\omega_0 \in B$, the sequence $\{\Upsilon^n \omega_0\}$ converges to v .

Theorem 1.7 ([9]). *Let (B, b, s) be a cb-ms, and let $\Upsilon : B \rightarrow B$ be a mapping such that:*

$$b(\Upsilon\omega, \Upsilon\varpi) \leq \alpha b(\omega, \varpi) + \beta b(\omega, \Upsilon\omega) + \gamma b(\varpi, \Upsilon\varpi)$$

for all $\omega, \varpi \in B$, where $\alpha, \beta, \gamma \in [0, 1)$ and $\alpha + \beta + \gamma < 1$. Then Υ admits a unique fp v , and for any starting point $\omega_0 \in B$, the sequence $\{\Upsilon^n \omega_0\}$ converges to v , provided that one of the following hold:

- Υ is continuous,
- $s\beta < 1$
- $s\gamma < 1$.

Theorem 1.8 ([5]). *Let (B, b, s) be a cb-ms, and let $\Upsilon : B \rightarrow B$ be a mapping such that:*

$$b(\Upsilon\omega, \Upsilon\varpi) \leq q \max \left\{ b(\omega, \varpi), b(\omega, \Upsilon\omega), b(\varpi, \Upsilon\varpi), \frac{1}{2}[b(\omega, \Upsilon\varpi) + b(\varpi, \Upsilon\omega)] \right\}$$

for all $\omega, \varpi \in B$, where $q \in [0, \frac{1}{s})$. Then Υ admits a unique fp v , and for any starting point $\omega_0 \in B$, the sequence $\{\Upsilon^n \omega_0\}$ converges to v , provided that Υ is continuous.

2. RESULT VIA IMPLICIT RELATION

In this section, we will first consider an implicit relation with some properties and give a few examples. Then, we will present a fp theorem in the b -ms, taking this relation into account.

Let $\mathcal{F}$ be the set of all functions $F : \mathbb{R}_+^6 \rightarrow \mathbb{R}$ satisfying the following conditions:

- (F1) $F(\xi_1, \dots, \xi_6)$ is nonincreasing in variables ξ_5 and ξ_6 .
(F2) there exists $h \in [0, 1)$ such that for $\eta > 0$

$$F(\eta, \mu, \mu, \eta, \eta + \mu, 0) \leq 0$$

or

$$F(\eta, \mu, \eta, \mu, 0, \eta + \mu) \leq 0$$

implies $\eta \leq h\mu$.

- (F3) $F(\eta, \eta, 0, 0, \eta, \eta) > 0$ for $\eta > 0$.

Also, for $s \geq 1$, let $\mathcal{F}_s$ be the set of all continuous functions $F \in \mathcal{F}$ satisfying the following conditions:

- (Fs1) $F(\xi_1, \dots, \xi_6)$ is nondecreasing in variable ξ_1 .
(Fs2) $F(\frac{\eta}{s}, 0, 0, \eta, \eta, 0) > 0$ or $F(\frac{\eta}{s}, 0, \eta, 0, 0, \eta) > 0$ for $\eta > 0$.

It is clear that $\mathcal{F}_s \subseteq \mathcal{F}$.

Example 2.1. *Let $F(\xi_1, \dots, \xi_6) = \xi_1 - \alpha\xi_2$, where $\alpha \in [0, 1)$. Then $F \in \mathcal{F}_s$ for all $s \geq 1$.*

Example 2.2. *Let $F(\xi_1, \dots, \xi_6) = \xi_1 - \alpha\xi_2 - \beta\xi_3 - \gamma\xi_4$, where $\alpha, \beta, \gamma \in [0, 1)$ and $\alpha + \beta + \gamma < 1$. Then $F \in \mathcal{F}$. Also if $s\beta < 1$ or $s\gamma < 1$ then $F \in \mathcal{F}_s$.*

Example 2.3. *Let $F(\xi_1, \dots, \xi_6) = \xi_1 - q \max \left\{ \xi_2, \xi_3, \xi_4, \frac{\xi_5 + \xi_6}{2} \right\}$, where $q \in [0, 1)$. Then $F \in \mathcal{F}$. Also if $qs < 1$, then $F \in \mathcal{F}_s$.*

Example 2.4. Let $F(\xi_1, \dots, \xi_6) = \xi_1 - c\{\xi_2 + |\xi_3 - \xi_4|\}$, where $c \in [0, 1)$. Then $F \in \mathcal{F}$. Also if $cs < 1$, then $F \in \mathcal{F}_s$.

Example 2.5. Let $F(\xi_1, \dots, \xi_6) = \xi_1 - \alpha \max\{\xi_2, \xi_3, \xi_4\} - \beta \frac{\xi_5 + \xi_6}{2}$, where $\alpha, \beta \in [0, 1)$ and $\alpha + \beta < 1$. Then $F \in \mathcal{F}$. Also if $(2\alpha + \beta)s < 2$, then $F \in \mathcal{F}_s$.

Theorem 2.1. Let (B, b, s) be a cb -ms, $\Upsilon : B \rightarrow B$ be a mapping satisfying:

$$F\left(b(\Upsilon\omega, \Upsilon\varpi), b(\omega, \varpi), b(\omega, \Upsilon\omega), b(\varpi, \Upsilon\varpi), \frac{b(\omega, \Upsilon\varpi)}{s}, \frac{b(\varpi, \Upsilon\omega)}{s}\right) \leq 0 \quad (2.1)$$

for all $\omega, \varpi \in B$, where $F \in \mathcal{F}$. Then Υ has a unique fp v , and for every $\omega_0 \in B$, the sequence $\{\Upsilon^n \omega_0\}$ converges to v provided that Υ is continuous.

Proof. Let $\omega_0 \in B$ be arbitrary. Define a sequence $\{\omega_n\}$ by $\omega_{n+1} = \Upsilon\omega_n$ for all $n \geq 0$. For the sake of brevity let $b_n = b(\omega_n, \omega_{n+1})$ for $n \geq 0$. If there exists $n_0 \in \mathbb{N}$ such that $b_{n_0} = 0$, then ω_{n_0} is a fp of Υ . In this case, the sequence $\{\omega_n\}$ will be an eventually constant sequence with the constant ω_{n_0} and will converge to ω_{n_0} . Now, let $b_n > 0$ for all $n \in \mathbb{N}$. Then from (2.1) we have,

$$F\left(b(\Upsilon\omega_n, \Upsilon\omega_{n+1}), b(\omega_n, \omega_{n+1}), b(\omega_n, \Upsilon\omega_n), b(\omega_{n+1}, \Upsilon\omega_{n+1}), \frac{1}{s}b(\omega_n, \Upsilon\omega_{n+1}), \frac{1}{s}b(\omega_{n+1}, \Upsilon\omega_n)\right) \leq 0$$

for all $n \in \mathbb{N}$. Thus, we have

$$F\left(b(\omega_{n+1}, \omega_{n+2}), b(\omega_n, \omega_{n+1}), b(\omega_n, \omega_{n+1}), b(\omega_{n+1}, \omega_{n+2}), \frac{b(\omega_n, \omega_{n+2})}{s}, 0\right) \leq 0$$

for all $n \in \mathbb{N}$. Now, from the last inequality and (F1) we have

$$F(b_{n+1}, b_n, b_n, b_{n+1}, b_n + b_{n+1}, 0) \leq 0$$

and hence from (F2), there exists $h \in [0, 1)$ such that

$$b_{n+1} \leq hb_n$$

or equivalently

$$b(\omega_{n+1}, \omega_{n+2}) \leq hb(\omega_n, \omega_{n+1})$$

for all $n \in \mathbb{N}$. From Lemma 1.5, we have $\{\omega_n\}$ is a Cauchy sequence in the cb -ms (B, b, s) . Hence there exists $v \in B$ such that $\lim \omega_n = v$, that is $\lim b(\omega_n, v) = 0$. Since Υ is continuous we have $\lim b(\omega_{n+1}, \Upsilon v) = \lim b(\Upsilon\omega_n, \Upsilon v) = 0$, and so by the uniqueness of the limit of $\{\omega_n\}$, we get $v = \Upsilon v$. Now, let w be also a fp of Υ with $b(v, w) > 0$, then from (2.1) we have

$$F\left(b(\Upsilon v, \Upsilon w), b(v, w), b(v, \Upsilon v), b(w, \Upsilon w), \frac{b(v, \Upsilon w)}{s}, \frac{b(w, \Upsilon v)}{s}\right) \leq 0$$

or equivalently we have

$$F\left(b(v, w), b(v, w), 0, 0, \frac{b(v, w)}{s}, \frac{b(w, v)}{s}\right) \leq 0.$$

Hence from (F1) we have

$$F(b(v, w), b(v, w), 0, 0, b(v, w), b(v, w)) \leq 0,$$

which is contradict to (F3). Hence, the fp of Υ is unique. $\square$

In Theorem 2.1, by considering the class $\mathcal{F}_s$, we can remove the continuity condition of Υ .

Theorem 2.2. *Let (B, b, s) be a cb-ms, $\Upsilon : B \rightarrow B$ be a mapping satisfying (2.1) for all $\omega, \varpi \in B$, where $F \in \mathcal{F}_s$. Then Υ has a unique fp v , and for every $\omega_0 \in B$, the sequence $\{\Upsilon^n \omega_0\}$ converges to v .*

Proof. As in the proof of Theorem 2.1, we can show that the iteration sequence $\{\omega_n\}$ is Cauchy and hence converges to some $v \in B$. Now, let $b(v, \Upsilon v) > 0$. Then, from (2.1) we have

$$F\left(b(\Upsilon \omega_n, \Upsilon v), b(\omega_n, v), b(\omega_n, \Upsilon \omega_n), b(v, \Upsilon v), \frac{b(\omega_n, \Upsilon v)}{s}, \frac{b(v, \Upsilon \omega_n)}{s}\right) \leq 0$$

or equivalently we have

$$F\left(b(\omega_{n+1}, \Upsilon v), b(\omega_n, v), b(\omega_n, \omega_{n+1}), b(v, \Upsilon v), \frac{b(\omega_n, \Upsilon v)}{s}, \frac{b(v, \omega_{n+1})}{s}\right) \leq 0.$$

Hence, by taking the continuity of F into account, from Lemma 1.2, (F1) and (Fs1) we have

$$F\left(\frac{b(v, \Upsilon v)}{s}, 0, 0, b(v, \Upsilon v), b(v, \Upsilon v), 0\right) \leq 0.$$

which is contradict to (Fs2). Hence we have $b(v, \Upsilon v) = 0$. The uniqueness follows from (F3). $\square$

If we consider special functions F in Theorem 2.1 or Theorem 2.2, we can obtain some fp theorems obtained in the literature, including Theorem 1.6, Theorem 1.7 and Theorem 1.8. Here we present two results obtained with Example 2.4 and Example 2.5.

Corollary 2.3. *Let (B, b, s) be a cb-ms, $\Upsilon : B \rightarrow B$ be a mapping satisfying:*

$$b(\Upsilon \omega, \Upsilon \varpi) \leq c \{b(\omega, \varpi) + |b(\omega, \Upsilon \omega) - b(\varpi, \Upsilon \varpi)|\}$$

for all $\omega, \varpi \in B$, where $c \in [0, 1)$. Then Υ has a unique fp v , and for every $\omega_0 \in B$, the sequence $\{\Upsilon^n \omega_0\}$ converges to v provided that one of the following hold:

- Υ is continuous,
- $sc < 1$.

Corollary 2.4. *Let (B, b, s) be a cb-ms, $\Upsilon : B \rightarrow B$ be a mapping satisfying:*

$$b(\Upsilon \omega, \Upsilon \varpi) \leq \alpha \max \{b(\omega, \varpi), b(\omega, \Upsilon \omega), b(\varpi, \Upsilon \varpi)\} + \beta \frac{b(\omega, \Upsilon \varpi) + b(\varpi, \Upsilon \omega)}{2s}$$

for all $\omega, \varpi \in B$, where $\alpha, \beta \in [0, 1)$ and $\alpha + \beta < 1$. Then Υ has a unique fp v , and for every $\omega_0 \in B$, the sequence $\{\Upsilon^n \omega_0\}$ converges to v provided that one of the following hold:

- Υ is continuous,
- $(2\alpha + \beta)s < 2$.

3. RESULTS FOR NEARLY LIPSCHITZIAN MAPPINGS

In this section, we will present the b -ms version of the notion of a nearly Lipschitzian mappings defined by Sahu [11] in the metric space and give a fp theorem for such mappings. Recall that a self mapping Υ of a b -ms (B, b, s) is said to be Lipschitz mapping if there exists $\lambda \geq 0$ such that

$$b(\Upsilon \omega, \Upsilon \varpi) \leq \lambda b(\omega, \varpi)$$

for all $\omega, \varpi \in B$. It is clear that every Lipschitz mapping is continuous.

Definition 3.1. Let (B, b, s) be a b -ms, $\Upsilon : B \rightarrow B$ be a mapping and $\{\alpha_n\}$ be a sequence in $\mathbb{R}_+$ such that $\alpha_n \rightarrow 0$. Then, Υ is said to be nearly Lipschitzian with respect to $\{\alpha_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $\lambda_n \geq 0$ such that

$$b(\Upsilon^n \omega, \Upsilon^n \varpi) \leq \lambda_n \{b(\omega, \varpi) + \alpha_n\} \quad (3.1)$$

for all $\omega, \varpi \in B$. The infimum of constants λ_n for which (3.1) holds is denoted by $\phi(\Upsilon^n)$ and called the nearly Lipschitz constant.

Example 3.1. Let $B = [0, 2]$ with the b -metric $b(\omega, \varpi) = (\omega - \varpi)^2$ and $\Upsilon : B \rightarrow B$ a mapping defined by

$$\Upsilon \omega = \begin{cases} \omega - \frac{1}{4} & \omega > 1 \\ \frac{1}{2} & \omega \leq 1 \end{cases}.$$

Since Υ is discontinuous, then it is non-Lipschitz mapping. However, by the routine calculation we can see

$$b(\Upsilon^n \omega, \Upsilon^n \varpi) \leq \frac{n}{2^n} \{b(\omega, \varpi) + \alpha_n\},$$

where

$$\alpha_n = \begin{cases} 16 & n \leq 4 \\ \frac{1}{n} & n > 4 \end{cases}.$$

Hence, Υ is nearly Lipschitzian mapping.

Theorem 3.1. Let (B, b, s) be a cb -ms and $\Upsilon : B \rightarrow B$ be a continuous nearly Lipschitzian mapping with nearly Lipschitz constant $\phi(\Upsilon^n)$. If

$$\limsup_{n \rightarrow \infty} \sqrt[n]{\phi(\Upsilon^n)} < 1, \quad (3.2)$$

then Υ has a unique fp v , and for every $\omega_0 \in B$, the sequence $\{\Upsilon^n \omega_0\}$ converges to v .

Proof. Let $\omega_0 \in B$ be arbitrary. Define a sequence $\{\omega_n\}$ by $\omega_{n+1} = \Upsilon \omega_n$ for all $n \geq 0$. For the sake of brevity let $b_n = b(\omega_n, \omega_{n+1})$ for $n \geq 0$. If there exists $n_0 \in \mathbb{N}$ such that $b_{n_0} = 0$, then ω_{n_0} is a fp of Υ . In this case, the sequence $\{\omega_n\}$ will be an eventually constant sequence with the constant ω_{n_0} and will converge to ω_{n_0} . Now, let $b_n > 0$ for all $n \in \mathbb{N}$. Hence

$$0 < b_n = b(\Upsilon^n \omega_0, \Upsilon^{n+1} \omega_0) \leq \phi(\Upsilon^n) \{b(\omega_0, \omega_1) + \alpha_n\} \quad (3.3)$$

for all $n \in \mathbb{N}$. Now, define $\limsup_{n \rightarrow \infty} \sqrt[n]{\phi(\Upsilon^n)} = \delta$, then from (3.2) and (3.3), we have $0 < \delta < 1$. Choose $\alpha \in (1, \frac{1}{\delta})$, then from (3.3) we have

$$\alpha^n b_n \leq \alpha^n \phi(\Upsilon^n) \{b(\omega_0, \omega_1) + M\}, \quad (3.4)$$

where $M = \sup \alpha_n$, for all $n \in \mathbb{N}$. Hence, from (3.4) we have

$$\sum_{n=1}^{\infty} \alpha^n b_n \leq \{b(\omega_0, \omega_1) + M\} \sum_{n=1}^{\infty} \alpha^n \phi(\Upsilon^n).$$

By the root test for the convergence of the series we have

$$\sum_{n=1}^{\infty} \alpha^n \phi(\Upsilon^n) < \infty$$

since

$$\limsup_{n \rightarrow \infty} \sqrt[n]{\alpha^n \phi(\Upsilon^n)} = \alpha \delta < 1.$$

Therefore, by the comparison test we have

$$\sum_{n=1}^{\infty} \alpha^n b_n = \sum_{n=1}^{\infty} \alpha^n b(\omega_n, \omega_{n+1}) < \infty,$$

and so from Lemma 1.4, $\{\omega_n\}$ is a Cauchy sequence in the cb -ms (B, b, s) . Hence there exists $v \in B$ such that $\lim \omega_n = v$, that is $\lim b(\omega_n, v) = 0$. Since Υ is continuous we have $\lim b(\omega_{n+1}, \Upsilon v) = \lim b(\Upsilon \omega_n, \Upsilon v) = 0$, and so by the uniqueness of the limit of $\{\omega_n\}$, we get $v = \Upsilon v$. Now, let w be also a fp of Υ with $b(v, w) > 0$, then from we have

$$\begin{aligned} \sum_{n=1}^{\infty} b(v, w) &= \sum_{n=1}^{\infty} b(\Upsilon^n v, \Upsilon^n w) \\ &\leq \sum_{n=1}^{\infty} \phi(\Upsilon^n) \{b(v, w) + \alpha_n\} \\ &\leq \{b(v, w) + M\} \sum_{n=1}^{\infty} \phi(\Upsilon^n) \\ &< \infty, \end{aligned}$$

which is a contradiction, since $\sum_{n=1}^{\infty} b(v, w) = \infty$. Hence, Υ has a unique fp. $\square$

4. CONCLUSION

In conclusion, this paper presents significant advancements in the theory of fixed point theorems in the context of b -metric spaces. The general result established in the first part of the paper not only provides a broader framework for fixed point theorems in b -metric spaces but also encompasses many existing results from the literature as special cases. By employing the implicit relation technique, we have extended the applicability of previous fixed point theorems to a more general setting. Furthermore, the introduction of the concept of nearly Lipschitzian mappings in b -metric spaces has led to the derivation of a new fixed point theorem, broadening the scope of mappings for which fixed points can be guaranteed. These contributions offer valuable insights and open up potential avenues for further research in the field of metric and b -metric spaces, particularly in the study of fixed point theory and its applications.

REFERENCES

- [1] A. Aghajani, M. Abbas, J. R. Roshan, *Common fixed point of generalized weak contractive mappings in partially ordered b -metric spaces*, *Mathematica Slovaca* **64**(4) (2014) 941–960.
- [2] M. Aslantaş, H. Sahin, U. Sadullah, *Some generalizations for mixed multivalued mappings*, *Applied General Topology* **23**(1) (2022) 169–178.
- [3] M. Aslantaş, H. Sahin, D. Turkoglu, *Some Caristi type fixed point theorems*, *The Journal of Analysis* **29**(1) (2021) 89–103.
- [4] M. Boriceanu, M. Bota, A. Petrusel, *Multivalued fractals in b -metric spaces*, *Central European Journal of Mathematics* **8**(2) (2010) 367–377.
- [5] M. F. Bota, L. Guran, A. Petrusel, *New fixed point theorems on b -metric spaces with applications to coupled fixed point theory*, *Journal of Fixed Point Theory and Applications*, **22** (2020) 74.
- [6] S. Czerwik, *Nonlinear set-valued contraction mappings in b -metric spaces*, *Atti del Seminario Matematico e Fisico dell'Università di Modena e Reggio Emilia* **46**(2) (1998) 263–276.
- [7] R. Miculescu, A. Mihail, *Caristi-Kirk type and Boyd&Wong-Browder-Matkowski-Rus type fixed point results in b -metric spaces*, *Filomat* **31**(14) (2017) 4331–4340.
- [8] R. Miculescu, A. Mihail, *New fixed point theorems for set-valued contractions in b -metric spaces*, *Journal of Fixed Point Theory and Applications* **19** (2017) 2153–2163.

- [9] Z. D. Mitrovic, *Fixed point results in b -metric spaces*, Fixed Point Theory **20**(2) (2019) 559–566.
- [10] H. Sahin, M. Aslantaş, A. A. Nasir Nasir, *Some extended results for multivalued F -contraction mappings*, Axioms **12**(2) (2023) 116.
- [11] D. R. Sahu, *Fixed points of demicontinuous nearly Lipschitzian mappings in Banach spaces*, Commentationes Mathematicae Universitatis Carolinae **46**(4) (2005) 653–666.
- [12] T. Suzuki, *Basic inequality on a b -metric space and its applications*, Journal of Inequalities and Applications **2017** (2017) 256.

ISHAK ALTUN,

DEPARTMENT OF MATHEMATICS, FACULTY OF ENGINEERING AND NATURAL SCIENCE, KIRIKKALE UNIVERSITY, 71450 YAHSIHAN, KIRIKKALE, TÜRKİYE

Email address: `ishakaltun@yahoo.com`

DİLDAR GÜLSEMIH AKEKİN,

DEPARTMENT OF MATHEMATICS, FACULTY OF ENGINEERING AND NATURAL SCIENCE, KIRIKKALE UNIVERSITY, 71450 YAHSIHAN, KIRIKKALE, TÜRKİYE

Email address: `dildar.17.ak@gmail.com`