

ABOUT G -CONVERGENCE AND G -SEQUENTIALLY CONVERGENCE

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ABSTRACT. In this paper we will discuss the similarities and differences for some topological properties in G -convergence and G -sequential convergence. In particular we will illustrate the ways in which these notions agree and are divided in some topological properties by looking at their definitions, characteristics, and implications. We will use examples from the literature [10] to distinguish between G -continuity and G -sequential continuity. Moreover, we will prove that these two kinds of continuity produce different outcomes and broaden the notion of the G -method in topological spaces.

1. INTRODUCTION

Continuity is essential to study different topological properties. In calculus and analysis, the idea of continuity is important because it forms the basis of numerous fundamental results. A deeper exploration of the nature of mathematical relationships and their applications in diverse domains is made possible through continuity of functions. Beyond theoretical mathematics, the concept of continuity have practical applications in fields like signal processing, engineering, and physics, where continuous models are necessary for accurate description and prediction of real-world problems. Sequential convergence is used in topology to describe the properties of spaces, especially in first-countable spaces where sequences can be used to characterize convergence. This idea gives the best concept for more intuitively analysing the behaviour of topological spaces by enabling mathematicians to explore limits, compactness, and continuity of functions. G -methods are first studied by Conner and Gross in [1], They studied that how changing the notion of convergence sequence affects the structure of the set of continuous functions. H.Çakallı extended this idea to topological groups and provided some interesting results that are useful for real cases [9]. Shou Lin and Li liu [2] studied more about the G -methods, G -sub methods and they extended their study to topological spaces. Then this idea is more studied by by H.Çakallı in the topological group setting, he also discussed the G -sequential compactness [3]. O. Mucuk and H.Çakallı continued their study for topological groups with operations in paper [4] where they demonstrated G -connectedness of them (see also [7, 8]). The concepts of G -sequential neighbourhoods and G -sequential continuity have recently been introduced

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by O.Mucuk and T. Şahan in [5]. Recently, G -convergent separation axioms are explored in article [11]. For a few G -methods and counterexamples, we refer readers to [10]; and [6] for a variety of G -convergence. Recently appeared papers [12] and [13] (see also [14]) are in the framework of this paper.

2. PRELIMINARIES

In this study, we examine how the structure of the set of continuous functions is changes between both types of convergences. Before start, it is important to define some definitions and notations for G -convergence and G -sequential convergences. In the paper $s(X)$ and $c(X)$ respectively indicate the sets of all sequences and convergent sequences in X . A map with a domain $c_G(X) \subset s(X)$ into X is referred to as a G -method of sequential convergence for X .

Definition 2.1. (1) A sequence $\mathbf{a} \in c_G(X)$ is G -convergent to a point ℓ when $G(\mathbf{a}) = \ell$. Specifically the function $\lim$ with domain $c(X)$ is a G -method whenever any convergence sequence has a unique limit.
 (2) Whenever any G -open neighbourhood of a point x includes eventually all terms, then a sequence $\mathbf{a} = (a_n)$ in X is G -sequentially convergent to the point $x \in X$ and denoted as $G_s(\mathbf{a}) = x$.

Note that in order to distinguish from G -convergence, we furthermore employ the word "sequentially" in this context.

Definition 2.2. (1) If a sequence $\mathbf{a} = (a_n) \in c_G(B)$ of the terms in subset $B \subseteq X$ is G -convergent to some point $x \in X$, then x is in the G -hull of B , and it is denoted as $[B]^G$ and if $[B]^G \subseteq B$ then it is called G -closed. A subset B , whose complement is G -closed, is G -open.
 (2) A point m provided that there exists a sequence $\mathbf{a} = (a_n)$ in a subset $B \subseteq X$ which is G -sequentially convergent to $m \in X$, is said to be in the G -sequentially hull of B and write $x \in [B]_G$. A subset B with $[B]_G \subseteq B$ is said to be G -sequentially closed and the complement of an open set is called G -sequentially open.
 (3) G -closure of a subset A is the intersection of G -closed subsets including A and denoted by $\overline{A}^G$.

It is a result of this definition that any G -open subset is G -sequentially open and therefore any G -closed subset is G -sequentially closed. Since we have difficulties to create G -sequentially open subsets we just consider the G -sequentially open subsets created by G -open subsets as finite intersections and any unions. That means by G -sequentially open subsets we understand open subset in the topology generated by the G -open subsets as a subbase.

Theorem 2.1. For a set X endowed with a G -sequential method and $C, D \subseteq X$, the statements below hold.

- (i) $[C \cap D]_G \subseteq [C]_G \cap [D]_G$.
- (ii) $[C \cup D]_G = [C]_G \cup [D]_G$.

The proof can be performed by using definition and therefore it is ignored.

Definition 2.3. Whenever any sequence $\mathbf{a} = (a_n)$ with G -sequentially convergence to l is nearly included in B , the point l is in the G -sequentially interior point of B and written as $l \in B_G^0$.

Here we remark that G -sequentially openness of a subset $B \subseteq X$ is equivalent to $B_G^0 \subseteq B$.

Theorem 2.2. For a set X endowed with a G -sequentially method and A, B subsets of X , we can state the following.

- (i) $(A \cap B)_G^0 = A_G^0 \cap B_G^0$.
- (ii) $A_G^0 \cup B_G^0 \subseteq (A \cup B)_G^0$.

Hence we can say that the intersection and union of G -sequentially open subsets are also G -sequentially open.

Example 2.1. [6] Let G be a method on $\mathbb{R}$ such that $G(\mathbf{x}) = \lim \frac{x_n + x_{n+1}}{2}$. Then G -closed subsets are $\mathbb{R}$, $\emptyset$, $[a, b]$, $(-\infty, a]$, $[b, \infty)$ and $\{a\}$ for $a, b \in \mathbb{R}$; and G -open subsets are the complements of these G -closed subsets: $\emptyset$, $\mathbb{R}$, $(-\infty, a) \cup (b, \infty)$, (a, ∞) , $(-\infty, b)$ and $\mathbb{R} \setminus \{a\}$ for $a, b \in \mathbb{R}$. The G -sequentially open subsets are those generated by G -open sets.

The sequence $\mathbf{a} = (a_n) = (1, 5, 1, 5, \dots)$ has G -convergence to 3 however it has no G -sequentially convergence to any point.

The sequence $\mathbf{a} = (a_n) = (1, 0, 2, 0, 3, 0, \dots)$ neither has G -convergence nor G -sequentially convergence to any point.

The sequence $\mathbf{a} = (a_n) = (\frac{1}{n})$ is both G -convergent and G -sequentially convergent to 0.

Example 2.2. Let $G(\mathbf{x}) = x_1$ for any sequence $\mathbf{x} = (x_n)$. Then any subset is both G -closed and G -open. That implies any subset is G -sequentially open and G -sequentially closed.

Example 2.3. Let G be the constant method given by $G(\mathbf{x}) = c$ for a constant $c \in X$, which means any sequence is G -convergent to c . Then $\emptyset$ and any subset containing c is G -closed and G -sequentially closed. Hence X and any subset not containing c is G -open and G -sequentially open. Hence any sequence has G -sequentially convergence to c , since only G -open neighbourhood of c is X . For $b \neq c$ any sequence $\mathbf{a} = (a_n)$ has G -sequentially convergence to b if and only if all terms are eventually b , since $\{b\}$ is a G -sequentially open neighbourhood of b .

3. G -CONTINUITY AND G -SEQUENTIAL CONTINUITY

- Definition 3.1.**
- (1) A map $f: X \rightarrow X$ is G -continuous if $f(\mathbf{x}) \in c_G(X)$ and $G(f(\mathbf{x})) = f(G(\mathbf{x}))$ whenever $\mathbf{x} \in c_G(X)$ [9].
 - (2) A map $g: X \rightarrow X$ is G -sequentially continuous if the sequence $g(\mathbf{x})$ has G -sequentially convergence to $g(a)$ whenever $\mathbf{x} = (x_n)$ has G -sequential convergence to a .

In below we give some examples of G -continuous and G -sequential continuous functions.

We emphasise that as a result of the definitions it can be proved that any G -continuous function is G -sequentially continuous.

Example 3.1. Consider G -method on $\mathbb{R}$ given in Example 2.1.

- (1) For the function $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$, for instance the sequence $\mathbf{x} = (x_n) = (0, 2, 0, 2, \dots)$ has G -convergence to 1 with $G(\mathbf{x}) = 1$ and $f(G(\mathbf{x})) = 1$ but $(f(\mathbf{x})) = (0, 4, 0, 4, \dots)$ which is G -converging to 2 with $G(f(\mathbf{x})) = 2$. Hence $f(G(\mathbf{x})) \neq G(f(\mathbf{x}))$ and therefore the function f is not G -continuous.

By evaluating G -sequentially convergences of the sequences or equivalently checking that the inverse images of G -sequentially open subsets are G -sequentially open, we can see that the function f is G -sequentially continuous.

- (2) One can check G -continuity of the function $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x + 1$. For a sequence $\mathbf{x} = (x_n)$ in $c_G(\mathbb{R})$ one has that $G(\mathbf{x}) = \lim \frac{x_n + x_{n+1}}{2}$ and therefore

$$\begin{aligned} G(f(\mathbf{x})) &= G(x_n + 1) \\ &= \lim \frac{(x_n + 1) + (x_{n+1} + 1)}{2} \\ &= \lim \frac{x_n + x_{n+1}}{2} + 1 \\ &= f(G(\mathbf{x})) \end{aligned}$$

which denotes G -continuity of the function f . Hence it is also G -sequentially continuous.

- (3) A constant map and identity map are G -sequentially continuous.

Example 3.2. Let a method G be defined as in Example 2.2. Then G -continuity and G -sequentially continuity are as follows:

Let $f: X \rightarrow X$ be a function. For any sequence $\mathbf{x} = (x_n)$ with G -convergence to x_1 , the sequence $(f(\mathbf{x}))$ has G -convergence to $f(x_1)$. Hence $f(G(\mathbf{x})) = G(f(\mathbf{x})) = f(x_1)$ which indicates that the function f is G -continuous.

Moreover f is G -sequentially continuous. Since in this method any subset is G -open, if a sequence $\mathbf{x} = (x_n)$ in X has G -sequentially convergence to a , then $\mathbf{x} = (x_1, x_2, \dots, x_{n_0}, a, a, \dots)$ and therefore $f(\mathbf{x}) = (f(x_1), f(x_2), \dots, f(x_{n_0}), f(a), f(a), \dots)$ which is G -sequentially convergent to $f(a)$.

Example 3.3. Let G be the constant method defined in Example 2.3. Then we have the following.

Let $g: X \rightarrow X$ be a function. Any sequence $\mathbf{a} = (a_n)$ is G -convergent to c and the image $(g(\mathbf{a}))$ is also G -convergent to c . Hence g is G -continuous when $g(c) = c$ and otherwise it is not G -continuous.

Remark. Here we note that if $f: X \rightarrow X$ is a G -continuous map, then $f(\overline{A}^G) \subseteq \overline{f(A)}^G$ for any subset $A \subseteq X$, but the wise versa of this statement is not true. For example in Example 3.1 (1) if $A = \{0, 1\}$, then $\overline{A}^G = [0, 1]$ and $f(\overline{A}^G) = [0, 1]$; and $f(A) = \{0, 1\}$ and $\overline{f(A)}^G = [0, 1]$. Hence $f(\overline{A}^G) = \overline{f(A)}^G = [0, 1]$ but the function $f(x) = x^2$ is not G -continuous. For the subset $A = \{-1, 1\}$, we have $\overline{A}^G = [-1, 1]$, $f(\overline{A}^G) = [0, 1]$, $f(A) = \{1\}$ and $\overline{f(A)}^G = \{1\}$. Thus $f(\overline{A}^G) \not\subseteq \overline{f(A)}^G$ and f is not G -continuous.

Remark. We note that if $f: X \rightarrow X$ is a G -continuous map, then $f([A]^G) \subseteq [f(A)]^G$ for any subset $A \subseteq X$, but as we can see in the example below the latter condition is not satisfied and therefore f is not G -continuous:

For the method in Example 3.1 (1) and the function $f(x) = x^2$, if $A = \{0, 1\}$, then $[A]^G = \{0, \frac{1}{2}, 1\}$ and $f([A]^G) = \{0, \frac{1}{4}, 1\}$. However $f(A) = \{0, 1\}$ and $[f(A)]^G = \{0, \frac{1}{2}, 1\}$. Hence we conclude that $[f(A)]^G \not\subseteq f([A]^G)$ and therefore f is not G -continuous.

4. G -OPEN AND G -SEQUENTIALLY OPEN FUNCTIONS

- Definition 4.1.** (1) A map $f: X \rightarrow X$ in which the image of any G -closed (resp. G -open) subset is also G -closed (resp. G -open), is called G -closed (resp. G -open).
 (2) A map $f: X \rightarrow X$ in which the image of any G -sequentially closed (resp. G -sequentially open) subset is also G -sequentially closed (resp. G -sequentially open), is called G -sequentially closed (resp. G -sequentially open).

Example 4.1. Consider the method in Example 2.1 and the map defined by

$$f(x) = \begin{cases} x + 1, & x \geq 0 \\ x, & x < 0 \end{cases}$$

- (1) For instance $A = [0, 1]$ is G -closed but $f(A) = \{0\} \cup (1, 2]$ is not G -closed and therefore the map f is not G -closed.
- (2) The subset $A = (-\infty, 1)$ is a G -open subset and $f(A) = (-\infty, 0] \cup (1, 2]$ is not a G -open subset; and therefore the map f is not G -open.

Example 4.2. For the method given in Example 2.2, any map $f: X \rightarrow X$ is G -closed and G -sequentially closed, since any subset is G -closed and G -sequentially closed. Moreover any map $f: X \rightarrow X$ is G -open and G -sequentially open, since any subset is G -open and G -sequentially open.

Example 4.3. For the constant method defined in Example 2.3 we check the following.

- (1) For this method a map $g: X \rightarrow X$ is G -closed whenever $g(c) = c$ since any subset containing c is G -closed. Same criterion is valid for G -sequential closed maps.
- (2) A subset B is G -open if $c \notin B$. Hence a function $g: X \rightarrow X$ is G -open if whenever $c \notin B$, then $c \notin g(B)$. Same criterion is valid for G -sequentially open maps.

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REFERENCES

- [1] J. Connor and K.-G. Grosse-Erdmann, *Sequential definitions of continuity for real functions*, Rocky Mountain J. Math. **33**(1) (2003), 93–121.
- [2] S. Lin and L. Liu, *G-methods, G-spaces and G-continuity in topological spaces*, Topology Appl. **212** (2016), 29–48.
- [3] H. Çakallı, *Sequential definitions of compactness*, Appl. Math. Lett. **21**(6) (2008), 594–598.
- [4] O. Mucuk and H. Çakallı, *G-sequentially connectedness for topological groups with operations*, Filomat. **32**(3) (2018), 1079–1089.
- [5] O. Mucuk and T. Şahan, *On G-sequential Continuity*, Filomat. **28**(6) (2014), 1181–1189.
- [6] S. Behram and O. Mucuk, *About varieties of G-sequential methods, G-hulls and G-closures*, Proceedings of International Mathematical Sciences. **5**(2) (2023), 81–86.
- [7] H. Çakallı, *Sequential definitions of connectedness*, Appl. Math. Lett. **25**(3) (2012), 461–465.
- [8] H. Çakallı, and O. Mucuk, *On connectedness via a sequential method*, Revista de la Unión Matemática Argentina. **54**(2) (2013), 101–109.
- [9] H. Çakallı, *On G-continuity*, Comput. Math. Appl., **61** (2011), 313–318.
- [10] O. Mucuk and S. Behram, *Counter examples of G- convergent method*, AIP Conf. Proc. **2879** (2022), 070001.
- [11] O. Mucuk, S. Behram and H. Çakallı, *G-convergently separation axioms*, Filomat, **38**(18) (2024), 6433–6441.
- [12] S. Behram and O. Mucuk, *G-convergence and G-sequential spaces*, Hacet. J. Math. Stat. **54**(6) (2025), 2326–2334.
- [13] L. Liu, S. Lin and X. Zhoua, *On Gs-convergence and Gs-countable compactness*, Topology Appl. **375** (2025), Paper No. 109554, 23 pp.
- [14] S. Behram, *G-convergence and G-sequential spaces for G-methods*, Ph.D Thesis, Erciyes University, (2025).

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