

A SPECTRAL FRACTIONAL HIROTA BILINEAR FORMULATION FOR A TIME-FRACTIONAL NONLINEAR SCHRÖDINGER FLOW: ONE-SOLITON SOLUTION AND OBSTRUCTION TO FINITE TWO-SOLITON TRUNCATION

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ABSTRACT. We define a skew-adjoint spectral time-fractional operator $\mathcal{D}_t^\alpha$ ($0 < \alpha \leq 1$) by the Fourier multiplier $\widehat{\mathcal{D}_t^\alpha \varphi}(\omega) = i \operatorname{sgn}(\omega) |\omega|^\alpha \hat{\varphi}(\omega)$. This provides a single-valued exponential Hirota calculus. Using the induced fractional Hirota D_t^α -operator we derive a bilinear system for a time-fractional focusing nonlinear Schrödinger (NLS) flow, recovering the classical Hirota form when $\alpha = 1$. A stationary bright one-soliton τ -function is obtained with dispersion relation $\omega = k^{2/\alpha}$. For $0 < \alpha < 1$ we prove that the classical finite two-soliton Hirota truncation is generically inconsistent: interaction-mode matching forces identities such as $(2\omega_2 - \omega_1)^\alpha = 2\omega_2^\alpha - \omega_1^\alpha$, contradicting strict concavity. Numerical residual plots illustrate the mismatch.

1. INTRODUCTION

The nonlinear Schrödinger (NLS) equation

$$iu_t + u_{xx} + 2|u|^2 u = 0 \quad (1.1)$$

is a canonical integrable model for nonlinear dispersive waves. Its soliton solutions can be constructed by the inverse scattering method or by Hirota's direct method, which rewrites (1.1) as a bilinear system for τ -functions [1–3]. Fractional-in-time generalizations of Schrödinger and NLS-type equations appear in models with memory and anomalous transport. In the classical fractional-derivative setting, the resulting evolution is typically nonlocal in time and the solution operator may lack semigroup structure [4, 5]. Recent work has also studied space–time fractional nonlinear Schrödinger equations from an analytical point of view [7]. However, incorporating time-fractional operators while preserving the algebraic closure properties underpinning soliton methods is subtle. In particular, complex fractional powers are multi-valued and the map $r \mapsto r^\alpha$ is non-additive for $0 < \alpha < 1$, both

2010 *Mathematics Subject Classification.* Primary 35Q55; Secondaries 35R11, 37K10.

Key words and phrases. time-fractional nonlinear Schrödinger equation; Hirota bilinear method; spectral fractional derivative; soliton; τ -function.

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Submitted on 18.01.2026. Accepted on 28.06.2026.

Communicated by Huseyin Cakalli.

of which interfere with the cancellation mechanisms that make Hirota expansions truncate.

The goal of this paper is twofold. First, we propose a time-fractional NLS flow directly at the bilinear level using a skew-adjoint spectral time-fractional operator $\mathcal{D}_t^\alpha$. This avoids branch ambiguities in the exponential calculus and leads to a clean bright one-soliton solution. Second, we provide a rigorous obstruction showing that the classical finite two-soliton Hirota truncation (polynomial τ -support) fails for $0 < \alpha < 1$ and generic distinct soliton frequencies. We then indicate how this failure forces additional mixed-frequency exponential modes and motivates a non-polynomial (infinite-series) τ -representation.

2. PRELIMINARIES: SPECTRAL FRACTIONAL HIROTA TIME OPERATOR

We begin by recalling the classical Hirota bilinear operators that underlie Hirota's direct method. Here $C^\infty(\mathbb{R}^2; \mathbb{C})$ denotes the space of all complex-valued functions $F : \mathbb{R}^2 \rightarrow \mathbb{C}$ such that, for every $m, n \in \mathbb{N}_0$, the mixed partial derivative $\partial_x^m \partial_t^n F$ exists and is continuous on $\mathbb{R}^2$. For smooth functions $f, g \in C^\infty(\mathbb{R}^2; \mathbb{C})$, with $f = f(x, t)$ and $g = g(x, t)$, the Hirota bilinear operators are

$$D_x^m D_t^n f \cdot g = (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n f(x, t) g(x', t')|_{x'=x, t'=t},$$

with $m, n \in \mathbb{N}_0$. For exponential phases $\theta_j = k_j x + \omega_j t + \delta_j$ one has

$$D_x^m D_t^n e^{\theta_1} \cdot e^{\theta_2} = (k_1 - k_2)^m (\omega_1 - \omega_2)^n e^{\theta_1 + \theta_2}. \quad (2.1)$$

Next we introduce the spectral time-fractional operator used in this work, chosen to avoid branch ambiguity in the exponential calculus. Let $0 < \alpha \leq 1$. To keep the exponential Hirota calculus single-valued while retaining a Fourier-multiplier interpretation, we use the odd purely imaginary symbol

$$\widehat{\mathcal{D}_t^\alpha \phi}(\omega) := i \operatorname{sgn}(\omega) |\omega|^\alpha \widehat{\phi}(\omega), \quad (2.2)$$

for functions $\phi(t)$ with Fourier transform $\widehat{\phi}(\omega) = \int_{\mathbb{R}} e^{-i\omega t} \phi(t) dt$. Note that $\mathcal{D}_t^\alpha$ has an odd purely imaginary Fourier symbol, whereas $i\mathcal{D}_t^\alpha$ has the real odd symbol $-\operatorname{sgn}(\omega)|\omega|^\alpha$, which is the combination appearing in the dispersive term of the fractional NLS flow. For $\alpha = 1$ this coincides with ∂_t since $i \operatorname{sgn}(\omega)|\omega| = i\omega$. The operator (2.2) can be viewed (formally) as the composition of the Hilbert transform and a Riesz-type fractional operator; in particular it is skew-adjoint on $L^2(\mathbb{R}_t)$. A related Fourier-multiplier construction of a spectral fractional Hirota bilinear operator, with an application to a time-fractional KdV equation, was studied in [8].

Define the signed power

$$r^{[\alpha]} := \operatorname{sgn}(r) |r|^\alpha, \quad r \in \mathbb{R}, \quad (\operatorname{sgn}(0) = 0),$$

We shall also use the standard terminology that a function $F : I \rightarrow \mathbb{R}$ is strictly concave on an interval I if, for all distinct $a, b \in I$ and all $0 < \lambda < 1$,

$$F(\lambda a + (1 - \lambda)b) > \lambda F(a) + (1 - \lambda)F(b).$$

Finally, set

$$D_t^\alpha e^{i\omega_1 t} \cdot e^{i\omega_2 t} := i(\omega_1 - \omega_2)^{[\alpha]} e^{i(\omega_1 + \omega_2)t}. \quad (2.3)$$

Remark. If one uses the principal-branch complex power $(i\omega)^\alpha$, then $(i\omega)^\alpha = |\omega|^\alpha e^{i\alpha\pi \operatorname{sgn}(\omega)/2}$ and the phase depends on α . The choice (2.2) fixes the phase to $\pm\pi/2$ for all α , which is convenient for the τ -function algebra and keeps dispersion relations real for the stationary bright soliton derived below.

3. A TIME-FRACTIONAL NLS FLOW AND ITS BILINEAR FORM

Fractional NLS equation. We consider the focusing time-fractional NLS-type equation

$$i\mathcal{D}_t^\alpha u + u_{xx} + 2|u|^2 u = 0, \quad 0 < \alpha \leq 1, \quad (3.1)$$

where $\mathcal{D}_t^\alpha$ is the Fourier multiplier (2.2). For $\alpha = 1$ we recover (1.1).

Fractional Hirota bilinear system. Introduce τ -functions $f, g, h \in C^\infty(\mathbb{R}^2; \mathbb{C})$, with $f = f(x, t)$, $g = g(x, t)$ and $h = h(x, t)$, via

$$u = \frac{g}{f}, \quad \bar{u} = \frac{h}{f},$$

on any region where $f(x, t) \neq 0$. For $\alpha = 1$ a standard bilinearization of (1.1) is [1]

$$\begin{cases} (iD_t + D_x^2)g \cdot f = 0, \\ (-iD_t + D_x^2)h \cdot f = 0, \\ D_x^2 f \cdot f = 2gh. \end{cases} \quad (3.2)$$

Motivated by (3.2) and (2.3), we define the fractional bilinear NLS system as

$$\begin{cases} (iD_t^\alpha + D_x^2)g \cdot f = 0, \\ (-iD_t^\alpha + D_x^2)h \cdot f = 0, \\ D_x^2 f \cdot f = 2gh, \end{cases} \quad (3.3)$$

with $0 < \alpha \leq 1$.

Remark. For $\alpha \neq 1$, the equivalence between (3.3) and (3.1) is understood in a Hirota-spectral sense: (3.3) is the primary structure and (3.1) is the induced nonlinear equation under the transformation $u = g/f$, $\bar{u} = h/f$. We do not claim a general fractional chain rule for $\mathcal{D}_t^\alpha$. All computations below are performed in the exponential τ -function class where D_t^α acts algebraically via (2.3).

4. BRIGHT ONE-SOLITON SOLUTION

We seek a bright one-soliton of (3.3) of the form

$$g = e^\theta, \quad h = e^{\bar{\theta}}, \quad f = 1 + e^{\theta + \bar{\theta} + A},$$

with phase $\theta \in C^\infty(\mathbb{R}^2; \mathbb{C})$ given by

$$\theta(x, t) = kx + i\omega t + \delta,$$

where $k > 0$, $\omega > 0$ and $\delta \in \mathbb{C}$. Using (2.1) and (2.3), the first bilinear equation in (3.3) evaluated on $(g, f) = (e^\theta, 1)$ yields

$$\omega^\alpha = k^2 \quad \Longleftrightarrow \quad \omega = k^{2/\alpha}. \quad (4.1)$$

The third equation gives

$$e^A = \frac{1}{(k + \bar{k})^2} = \frac{1}{4k^2}.$$

Hence

$$u(x, t) = k \operatorname{sech}(kx + x_0) e^{i(\omega t + \phi_0)}, \quad \omega = k^{2/\alpha}, \quad (4.2)$$

where $x_0 \in \mathbb{R}$ and $\phi_0 \in \mathbb{R}$ are determined by δ . As $\alpha \rightarrow 1$ we recover the classical stationary bright soliton of (1.1).

5. TWO-SOLITON HIROTA TRUNCATION AND A RIGOROUS OBSTRUCTION FOR
 $0 < \alpha < 1$

For the classical NLS ($\alpha = 1$), a two-soliton solution can be obtained by a finite Hirota expansion (polynomial τ -support). We show that the same finite truncation is incompatible with (3.3) when $0 < \alpha < 1$, except in a degenerate equal-frequency case.

Following the standard Hirota perturbation scheme [1, 2], introduce a bookkeeping parameter ε and expand

$$g = \varepsilon g_1 + \varepsilon^3 g_3, \quad h = \varepsilon h_1 + \varepsilon^3 h_3, \quad f = 1 + \varepsilon^2 f_2 + \varepsilon^4 f_4. \quad (5.1)$$

For two solitons set

$$g_1 = e^{\theta_1} + e^{\theta_2}, \quad h_1 = e^{\bar{\theta}_1} + e^{\bar{\theta}_2}, \quad \theta_j = k_j x + i\omega_j t + \delta_j,$$

with $k_j > 0$ and $\omega_j > 0$. The classical truncation assumes that f_2 contains only the mixed quadratic modes and that g_3, h_3 contain only the cubic modes:

$$f_2 = c_{11}e^{\theta_1+\bar{\theta}_1} + c_{22}e^{\theta_2+\bar{\theta}_2} + c_{12}e^{\theta_1+\bar{\theta}_2} + c_{21}e^{\theta_2+\bar{\theta}_1}, \quad (5.2)$$

$$g_3 = a_{12}e^{\theta_1+\theta_2+\bar{\theta}_1} + a_{21}e^{\theta_1+\theta_2+\bar{\theta}_2},$$

$$h_3 = \bar{a}_{12}e^{\bar{\theta}_1+\bar{\theta}_2+\theta_1} + \bar{a}_{21}e^{\bar{\theta}_1+\bar{\theta}_2+\theta_2},$$

$$f_4 = d e^{\theta_1+\theta_2+\bar{\theta}_1+\bar{\theta}_2}. \quad (5.3)$$

Lemma 5.1. *Let $0 < \alpha < 1$. Then $r \mapsto r^\alpha$ is strictly concave on $(0, \infty)$. In particular, for any $a \neq b$ in $(0, \infty)$,*

$$(2b - a)^\alpha \neq 2b^\alpha - a^\alpha \quad \text{whenever } 2b - a > 0. \quad (5.4)$$

Proof. The strict concavity of r^α on $(0, \infty)$ for $0 < \alpha < 1$ is standard. Applying concavity with weights $1/2$ to the pair $(a, 2b - a)$ (whose average is b) yields $b^\alpha > \frac{1}{2}(a^\alpha + (2b - a)^\alpha)$ when $a \neq b$, which rearranges to (5.4). $\square$

Proposition 5.2. *Fix $0 < \alpha < 1$. Assume the classical two-soliton truncation (5.1)–(5.3) with $\omega_j > 0$ and the (fractional) dispersion relations*

$$\omega_j^\alpha = k_j^2, \quad j = 1, 2. \quad (5.5)$$

Then the fractional bilinear system (3.3) admits such a truncation only in the degenerate case $\omega_1 = \omega_2$. In particular, for generic distinct soliton frequencies $\omega_1 \neq \omega_2$, the classical finite two-soliton τ -support is incompatible with (3.3).

Proof. We match coefficients of linearly independent exponential modes produced by (3.3) under the ansatz (5.1)–(5.3).

Step 1: At order ε in $(iD_t^\alpha + D_x^2)g \cdot f = 0$, the modes e^{θ_j} yield

$$\omega_j^{[\alpha]} = k_j^2 \quad (j = 1, 2).$$

Since $\omega_j > 0$, this is (5.5). At order ε^2 in $D_x^2 f \cdot f = 2gh$, matching the four mixed modes in (5.2) gives

$$c_{11} = \frac{1}{4k_1^2}, \quad c_{22} = \frac{1}{4k_2^2}, \quad c_{12} = c_{21} = \frac{1}{(k_1 + k_2)^2}.$$

Step 2: At order ε^3 in $(iD_t^\alpha + D_x^2)g \cdot f = 0$, consider the mode $e^{\theta_1 + \theta_2 + \bar{\theta}_2}$. It receives contributions from the pairings $e^{\theta_2} \cdot c_{12}e^{\theta_1 + \bar{\theta}_2}$ and $a_{21}e^{\theta_1 + \theta_2 + \bar{\theta}_2} \cdot 1$. Using (2.3) and (2.1) one obtains the explicit relation

$$a_{21} = \frac{k_1^3 - k_1 k_2^2 - k_2^3 + k_2(2\omega_2 - \omega_1)^{[\alpha]}}{4k_2^2(k_1 + k_2)^3}. \quad (5.6)$$

Similarly, matching the mode $e^{\theta_1 + \theta_2 + \bar{\theta}_1}$ yields

$$a_{12} = \frac{-k_1^3 - k_1^2 k_2 + k_1(2\omega_1 - \omega_2)^{[\alpha]} + k_2^3}{4k_1^2(k_1 + k_2)^3}. \quad (5.7)$$

Step 3: Next, at order ε^4 in $D_x^2 f \cdot f = 2gh$, the cubic terms in g and h force a_{12}, a_{21} to satisfy the purely spatial identities

$$a_{12} = \frac{(k_1 - k_2)^2}{4k_1^2(k_1 + k_2)^2}, \quad a_{21} = \frac{(k_1 - k_2)^2}{4k_2^2(k_1 + k_2)^2}. \quad (5.8)$$

Step 4: Equating the two expressions for a_{21} in (5.6) and (5.8) and using (5.5) (so $k_j^2 = \omega_j^\alpha$) yields

$$(2\omega_2 - \omega_1)^{[\alpha]} = 2\omega_2^{[\alpha]} - \omega_1^{[\alpha]}. \quad (5.9)$$

Assume $\omega_1 \neq \omega_2$ and, relabeling if necessary, take $\omega_1 < \omega_2$, so that $2\omega_2 - \omega_1 > 0$ and $\omega_j^{[\alpha]} = \omega_j^\alpha$. Then (5.9) becomes

$$(2\omega_2 - \omega_1)^\alpha = 2\omega_2^\alpha - \omega_1^\alpha. \quad (5.10)$$

By Lemma 5.1, (5.10) is impossible for $0 < \alpha < 1$ unless $\omega_1 = \omega_2$. This proves the claim. $\square$

Forced extra modes and a non-truncating Hirota series. Proposition 5.2 identifies the mechanism that breaks truncation: the bilinear difference structure produces mixed-frequency combinations such as $(2\omega_2 - \omega_1)^{[\alpha]}$. For $\alpha = 1$ these are linear and the compatibility conditions collapse to identities. For $0 < \alpha < 1$, non-additivity of $r \mapsto r^{[\alpha]}$ prevents this collapse and forces additional exponential modes beyond the classical finite set. One natural way to proceed is to abandon finite polynomial τ -support and to consider the full formal Hirota series

$$g = \sum_{n \geq 0} \varepsilon^{2n+1} g_{2n+1}, \quad h = \sum_{n \geq 0} \varepsilon^{2n+1} h_{2n+1}, \quad f = 1 + \sum_{n \geq 1} \varepsilon^{2n} f_{2n},$$

where each coefficient is a finite sum of exponentials built from $\{\theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2\}$. The bilinear system (3.3) generates at each order an inhomogeneous linear algebraic system for the new coefficients. In the classical case, these systems become homogeneous beyond a finite order and the series truncates. For $0 < \alpha < 1$, Proposition 5.2 shows that the forcing already fails to vanish at the first nontrivial interaction order unless extra modes are introduced, and hence the recursion does not truncate. We illustrate this numerically in Section 6.

6. NUMERICAL ILLUSTRATIONS

We present two simple plots highlighting the α -dependence of the soliton dispersion and the two-soliton mismatch.

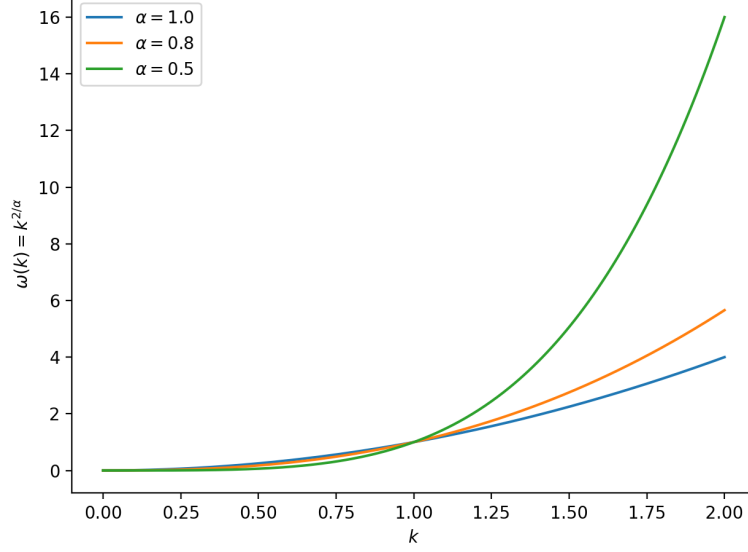


FIGURE 1. Dispersion relation $\omega(k) = k^{2/\alpha}$ from (4.1). The stationary envelope $k \operatorname{sech}(kx + x_0)$ is unchanged; only the temporal frequency depends on α .

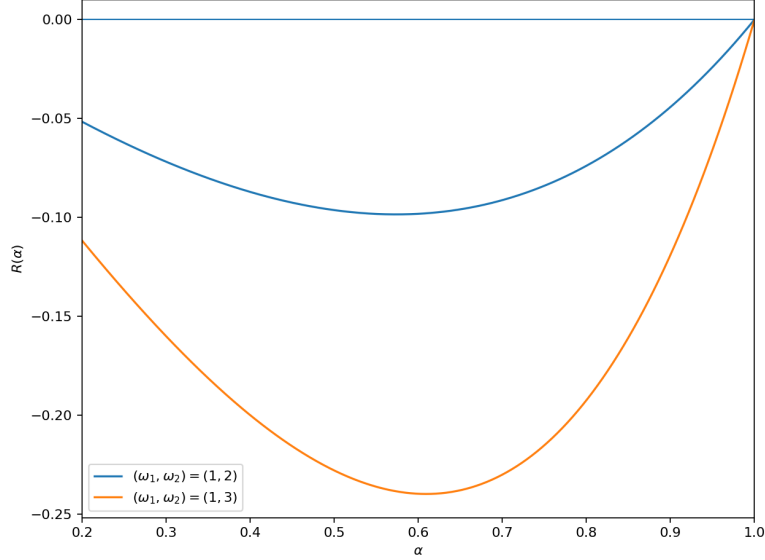


FIGURE 2. Residual $R(\alpha) = (2\omega_2 - \omega_1)^\alpha - (2\omega_2^\alpha - \omega_1^\alpha)$ for $(\omega_1, \omega_2) = (1, 2)$ and $(1, 3)$. Nonzero values for $0 < \alpha < 1$ illustrate the concavity obstruction.

7. CONCLUSION

The key modelling choice is the skew-adjoint spectral time-fractional operator $\widehat{\mathcal{D}_t^\alpha \varphi}(\omega) = i \operatorname{sgn}(\omega) |\omega|^\alpha \hat{\varphi}(\omega)$, so that $i\mathcal{D}_t^\alpha$ has the *real*, odd symbol $-\operatorname{sgn}(\omega) |\omega|^\alpha$. This choice keeps the exponential Hirota rules single-valued (in contrast to $(i\omega)^\alpha$ on a complex branch) and makes transparent why mixed frequency combinations prevent the classical finite two-soliton truncation for $0 < \alpha < 1$.

Within the resulting fractional Hirota framework we obtained a closed stationary bright one-soliton τ -function and the fractional dispersion relation $\omega = k^{2/\alpha}$. More importantly, we proved that the usual finite two-soliton Hirota polynomial support fails generically when $0 < \alpha < 1$: the fractional bilinear difference operator produces mixed-frequency expressions (e.g. $(2\omega_2 - \omega_1)^\alpha$) that cannot satisfy the compatibility identities required for truncation due to strict concavity of r^α . This obstruction indicates that genuinely interacting multi-soliton dynamics, if present for the spectral time-fractional flow, must involve enlarged τ -support, such as non-polynomial or infinite-series τ -representations.

Natural directions for further work include developing determinant/Gram-type τ -functions adapted to the spectral fractional symbol, and investigating the corresponding analytical questions for the induced nonlinear evolution, such as well-posedness and stability of the one-soliton family under $i\mathcal{D}_t^\alpha$ -dynamics.

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