



Ordered j -Approximation Spaces with an Application

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Abstract

In 1982, Pawlak introduced the concept of rough sets as a mathematical framework for addressing real-world problems. However, the dependence on equivalence relations within this framework has often proven inadequate for handling more complex cases. The aim of this study is to develop a more effective approach for such situations. In particular, partial order relations are combined with j -neighborhoods to capture not only vague similarities among objects but also hierarchical dependencies and priority structures that frequently arise in real data. For this purpose, ordered j -neighborhood classes are defined by employing increasing and decreasing sets based on partial order relations together with j -neighborhoods. The fundamental properties of these classes, as well as their relationships with j -neighborhoods and the increasing and decreasing equivalence classes, are then examined under various binary relations. Building on these classes, ordered j -approximation spaces are introduced as a natural generalization of both ordered approximation spaces and Pawlak approximation spaces. The findings show that the proposed approximations yield more capable results in reducing boundary regions compared to some existing approaches. Moreover, generalized nano topologies are constructed from these new structures, and their applicability is demonstrated through a real-world example.

Keywords: nano topological spaces; rough sets; ordered approximation spaces; ordered j -neighborhood classes; ordered j -approximation spaces

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1. Introduction

In computer science, and particularly within the field of artificial intelligence, the interpretation of knowledge under uncertain and imprecise conditions poses significant challenges. While classical set theory provides a mathematical framework for modeling precise and well-defined situations, it remains insufficient when dealing with vagueness and uncertainty. To address these limitations, several alternative set-theoretic approaches have been introduced, such as fuzzy sets, rough sets, and soft sets, which offer more flexible tools for mathematically representing and analyzing imprecision and uncertainty.

Rough set theory was proposed by Pawlak in 1982 [15] and has been widely studied in applications such as data analysis, attribute reduction, and decision rule generation. This theory classifies elements based on the indiscernibility relation, which partitions them into equivalence classes. The fundamental notions of rough set theory can be summarized as follows:

1. Lower approximation: the set of elements that definitely belong to a set.
2. Upper approximation: the set of elements that possibly belong to a set.
3. Boundary: the set difference between the lower and upper approximations. If this region is empty, the set is referred to as an exact set; otherwise, it is called a rough set.
4. Accuracy: the measure that indicates to what extent the elements of a set can be precisely defined in the information system, expressed as the ratio of the lower approximation to the upper approximation.

Since Pawlak's approximation spaces are based on equivalence relations, this becomes a limitation in modeling real life problems. In this direction, Yao [19, 20] introduced the notions of right and left neighborhoods using binary relations instead of equivalence relations, thereby pioneering the studies that generalized Pawlak's approximation spaces by neighborhoods. Although these efforts removed the restrictions of equivalence relations, the necessary conditions for preserving the fundamental properties of Pawlak's approximation spaces have been a research topic of the following studies. For example, Allam et al. [3, 4] investigated minimal right and left neighborhoods and showed that, for reflexive relations, presented approximation based on these neighborhoods preserves the properties of Pawlak's approximation spaces.

Abu Donia [2] extended Yao's right neighborhood based approximations and proposed generalized approximations through the intersection of right neighborhoods for a finite family of relations. Abd El-Monsef et al. [1] introduced the concept of j -neighborhoods, which involve also the intersections and unions of right, left, minimal right, and minimal left neighborhoods, and studied the properties of the topological structures defined by these neighborhoods. Atef et al. [9] defined approximation spaces based on j -neighborhoods and investigated their properties in detail. Subsequently, these studies on generalized approximation spaces were further enriched with new types of neighborhoods defined using j -neighborhoods. It has been observed that generalized approximations based on j -adhesion neighborhoods, introduced by Mareay [14] and Atef et al. [9], provide higher accuracy and reduce the boundary compared to Yao's approximation spaces. Some corrections to the work of Atef et al. [9] were made by El-Bably et al. [12]. Later, El-Bably and Abo-Tabl [10], as well as El-Bably and Al-Shami [11], proposed new generalized approximation spaces based on j -adhesion neighborhoods. In addition, based on the inclusion relation of j -neighborhoods, Al-Shami [5] introduced containment neighborhoods, while Al-Shami and Ciucci [6] introduced subset neighborhoods, and both presented generalized approximation spaces defined by these structures. Recent works on overlapping neighborhoods based on these neighborhoods are presented in [7, 8]. On the other hand, some of these studies also revealed the connections between rough set theory and topology, thereby contributing to the literature from a topological perspective. Recently, the ordered approximation spaces introduced by Shalil et al. [17] differ from the previous studies in that they are defined by using both an equivalence relation and a partial order relation, that is, an approximation space with two defining relations. They showed that if the partial order relation is replaced with the equality relation, these spaces coincide with Pawlak's.

The introduction of nano topology, derived from the fundamental components of rough set theory and based on the equivalence relation, by Thivagar and Richard [18], has provided a new perspective for applications of rough set theory. One of the main advantages of this concept is its ability to identify core attributes by reducing the number of attributes required in decision-making or diagnostic processes within an information system. By determining and focusing on the essential and significant attributes, it enables savings in time and resources. Core attributes, also known as irreducible attributes, are those whose removal changes the basis of the nano topology initially obtained from the information system. In contrast, reducible attributes are those whose removal does not change the basis. Logically, this implies that not all attributes in an information system are necessary, nor are all decision criteria dependent on every single attribute. Therefore, core attributes make the information system more efficient and reliable, thereby improving the accuracy of decision-making and problem-solving processes. Similarly, researchers [10, 13] have enriched the application scope by introducing the concept of generalized nano topology, defined under the conditions that must be satisfied by an arbitrary relation.

To generalize the ordered approximation spaces, we replace equivalence relations with j -neighborhoods. As a natural consequence of this framework, the resulting generalization extends Pawlak's model by simultaneously incorporating partial order relations and j -neighborhoods. The joint use of these two structures constitutes the main motivation of this study: while j -neighborhoods capture flexible and vague similarities among objects, partial order relations incorporate hierarchical and priority based dependencies into the model. In this way, both similarity and hierarchy present in real-world data can be represented simultaneously, leading to a more effective reduction of boundary and more reliable decision-making processes. In addition, the generalized nano topologies derived from novel generalizations offer a more effective tool for identifying core attributes in information systems.

This paper is structured as follows: first, the fundamental concepts are introduced. Then, the increasing and decreasing equivalence classes defined by Shalil et al. [17] are generalized via j -neighborhoods to obtain ordered j -neighborhood classes, and their properties are examined. Based on these classes, ordered j -approximation spaces are constructed, and their relationships with the second type of approximation spaces [9, 20] are analyzed. The properties of them are also discussed and supported by counterexamples. Subsequently, generalized nano topologies based on ordered j -approximation spaces are defined, and their effectiveness is demonstrated through examples. Finally, the proposed method is applied to an information system for identifying significant symptoms of lung cancer, highlighting both the theoretical and practical contributions.

2. Preliminaries

In this section, fundamental notions concerning rough sets and their different extensions will be reviewed.

Throughout this paper, the ordered pair $(\Sigma, \S)$ denotes an approximation space, where Σ is a nonempty finite set and $\S$ is a binary relation on Σ , unless stated otherwise. Specifically, when $\S$ is an equivalence relation, $(\Sigma, \S)$ is referred to as a Pawlak approximation space.

Definition 2.1. A binary relation $\S$ on Σ is said to be:

- i. reflexive relation if $x\S x$ for all $x \in \Sigma$,
- ii. symmetric relation if $x\S y$ then $y\S x$ for all $x, y \in \Sigma$,
- iii. transitive relation if $x\S y$ and $y\S z$ then $x\S z$ for all $x, y, z \in \Sigma$,
- iv. equivalence relation if it is reflexive, symmetric and transitive,
- v. antisymmetric relation if $x\S y$ and $y\S x$ then $x = y$ for all $x, y \in \Sigma$,
- vi. partial order relation if it is reflexive, antisymmetric and transitive,
- vii. equality relation if it is defined as $\Delta = \{(x, x) : x \in \Sigma\}$,
- viii. serial relation if there exists a $y \in \Sigma$ such that $x\S y$ for all $x \in \Sigma$,
- ix. inverse serial relation if there exists a $y \in \Sigma$ such that $y\S x$ for all $x \in \Sigma$.

Pawlak used the equivalence relation to define the lower and upper approximation operators, which are key concepts in rough set theory.

Definition 2.2. [15, 16] Let $\S$ be an equivalence relation on Σ and let $[x]_{\S} = \{y : x\S y\}$ be the equivalence class of $x \in \Sigma$ and the set of all equivalence classes is denoted by $\Sigma/\S$. The lower approximation $\underline{\S}$ and the upper approximation $\overline{\S}$ of a subset Y of Σ are defined, respectively, as follows.

$$\underline{\S}(Y) = \cup\{[x]_{\S} \in \Sigma/\S : [x]_{\S} \subseteq Y\} = \{x \in \Sigma : [x]_{\S} \subseteq Y\}$$

$$\overline{\S}(Y) = \cup\{[x]_{\S} \in \Sigma/\S : [x]_{\S} \cap Y \neq \emptyset\} = \{x \in \Sigma : [x]_{\S} \cap Y \neq \emptyset\}$$

Nevertheless, most actual issues only sometimes allow for the application of the equivalence relation and this constraint further reduces the extensive range of applications possible for rough set theory. To overcome these constraints, Yao generalized the Pawlak approximation operators by replacing equivalence relations with arbitrary binary relations.

Definition 2.3. [19, 20] Let $(\Sigma, \S)$ be an approximation space. The right neighborhood of $x \in \Sigma$, denoted by $\mathcal{N}_r(x)$, is defined as follows.

$$\mathcal{N}_r(x) = \{y \in \Sigma : x \S y\}$$

Definition 2.4. [19, 20] Let $(\Sigma, \S)$ be an approximation space. The right lower and upper approximations of $\Upsilon \subseteq \Sigma$ are given, respectively, as follows.

$$\underline{\mathcal{N}}_r(\Upsilon) = \{x \in \Sigma : \mathcal{N}_r(x) \subseteq \Upsilon\}$$

$$\overline{\mathcal{N}}_r(\Upsilon) = \{x \in \Sigma : \mathcal{N}_r(x) \cap \Upsilon \neq \emptyset\}$$

Then, Yao extended this definition by using the right neighborhood at the lower approximation operator and defining the upper approximation operator with duality.

Definition 2.5. [20] Let $(\Sigma, \S)$ be an approximation space. The second type of right lower and upper approximations of $\Upsilon \subseteq \Sigma$ are given, respectively, as follows.

$$\mathcal{N}_\S(\Upsilon) = \cup\{\mathcal{N}_r(x) : \mathcal{N}_r(x) \subseteq \Upsilon\}$$

$$\mathcal{N}_r^\S(\Upsilon) = [\mathcal{N}_\S(\Upsilon^c)]^c$$

In addition to the right neighborhoods, various neighborhood types were defined utilizing arbitrary binary relations.

Definition 2.6. [1, 3, 4, 19, 20] Let $(\Sigma, \S)$ be an approximation space. The j -neighborhoods of $x \in \Sigma$, denoted by $\mathcal{N}_j(x)$, are defined as follows for each $j \in \{l, i, u, < r >, < l >, < i >, < u >\}$.

1. $\mathcal{N}_l(x) = \{y \in \Sigma : y \S x\}$
2. $\mathcal{N}_i(x) = \mathcal{N}_r(x) \cap \mathcal{N}_l(x)$
3. $\mathcal{N}_u(x) = \mathcal{N}_r(x) \cup \mathcal{N}_l(x)$
4. $\mathcal{N}_{< r >}(x) = \begin{cases} \bigcap_{x \in \mathcal{N}_r(y)} \mathcal{N}_r(y) & , \text{there exists } \mathcal{N}_r(y) \text{ containing } x \\ \emptyset & , \text{otherwise} \end{cases}$
5. $\mathcal{N}_{< l >}(x) = \begin{cases} \bigcap_{x \in \mathcal{N}_l(y)} \mathcal{N}_l(y) & , \text{there exists } \mathcal{N}_l(y) \text{ containing } x \\ \emptyset & , \text{otherwise} \end{cases}$
6. $\mathcal{N}_{< i >}(x) = \mathcal{N}_{< r >}(x) \cap \mathcal{N}_{< l >}(x)$
7. $\mathcal{N}_{< u >}(x) = \mathcal{N}_{< r >}(x) \cup \mathcal{N}_{< l >}(x)$

In here, the concept of j -neighborhood refers to left, intersection, union, minimal right, minimal left, minimal intersection, and minimal union neighborhoods, respectively.

Definition 2.7. [1] Let $\S$ be any binary operation on Σ and ϕ_j be a map from Σ to 2^Σ which assigned each $x \in \Sigma$ with its j -neighborhood in 2^Σ . The triple $(\Sigma, \S, \phi_j)$ is called a j -neighborhood space (briefly, $j - \mathcal{N}\mathcal{S}$).

Atef et al. [9] defined generalized lower and upper approximation operators based on j -neighborhoods for $j \in \mathbb{T}$ using the technique in Definition 2.4.

Definition 2.8. [9] Let $(\Sigma, \S, \phi_j)$ be a $j - \mathcal{N}\mathcal{S}$. The j -lower and the j -upper approximations of $\Upsilon \subseteq \Sigma$ are defined, respectively, as follows for each $j \in \mathbb{T}$.

$$\underline{\mathcal{N}}_j(\Upsilon) = \{x \in \Sigma : \mathcal{N}_j(x) \subseteq \Upsilon\}$$

$$\overline{\mathcal{N}}_j(\Upsilon) = \{x \in \Sigma : \mathcal{N}_j(x) \cap \Upsilon \neq \emptyset\}$$

Additionally, Atef et al. [9] defined the second type of j -lower and j -upper approximation operators with j -neighborhoods as in Definition 2.5.

Definition 2.9. [9] Let $(\Sigma, \S, \phi_j)$ be a $j - \mathcal{N}\mathcal{S}$. The second type of j -lower and j -upper approximations of $\Upsilon \subseteq \Sigma$ are defined, respectively, as follows for each $j \in \mathbb{T}$.

$$\mathcal{N}_\S(\Upsilon) = \cup\{\mathcal{N}_j(x) : \mathcal{N}_j(x) \subseteq \Upsilon\}$$

$$\mathcal{N}_j^\S(\Upsilon) = [\mathcal{N}_\S(\Upsilon^c)]^c$$

Also, the j -neighborhood accuracy measure and the j -neighborhood boundary of $\Upsilon \subseteq \Sigma$ are, respectively, defined as $\alpha_j(\Upsilon) = \frac{|\mathcal{N}_\S(\Upsilon)|}{|\mathcal{N}_j^\S(\Upsilon)|}$ and

$$\mathcal{B}_j(\Upsilon) = \mathcal{N}_j^\S(\Upsilon) \setminus \mathcal{N}_\S(\Upsilon).$$

Shalil et al. [17] defined an ordered approximation space, a generalization of the Pawlak approximation space built by two specific relations that are the equivalence relation and the partial order relation.

Definition 2.10. [17] Let $\S$ be an equivalence relation, and let $\preceq$ be a partial order relation on Σ . For each $x \in \Sigma$, the increasing and decreasing equivalence classes are defined, respectively, as follows.

$$\mathcal{IE}(x) = \cup\{i_{\preceq}(y) : y \in \Sigma, x \S y\}$$

$$\mathcal{DE}(x) = \cup\{d_{\preceq}(y) : y \in \Sigma, x \S y\}$$

where the increasing set of $y \in \Sigma$ is $i_{\preceq}(y) = \{z \in \Sigma : y \preceq z\}$, and the decreasing set of $y \in \Sigma$ is $d_{\preceq}(y) = \{z \in \Sigma : z \preceq y\}$. Also an ordered approximation space (briefly, $\mathcal{OAS}$) is denoted by $(\Sigma, \S, \preceq)$.

Definition 2.11. [17] Let $(\Sigma, \S, \preceq)$ be an $\mathcal{OAS}$, and let $\Upsilon \subseteq \Sigma$. The following notions are defined, where $[x]_{\S} = \{y \in \Sigma : x \S y\}$ denotes the equivalence class of $x \in \Sigma$.

(1) The increasing and decreasing lower approximations of Υ are, respectively, defined as follows.

$$IL_{\S}(\Upsilon) = \cup\{\mathcal{IE}(x) : [x]_{\S} \subseteq \Upsilon\} \cap \Upsilon$$

$$DL_{\S}(\Upsilon) = \cup\{\mathcal{DE}(x) : [x]_{\S} \subseteq \Upsilon\} \cap \Upsilon$$

(2) The increasing and decreasing upper approximations of Υ are, respectively, defined by $IU_{\S}(\Upsilon) = [DL_{\S}(\Upsilon^c)]^c$ and $DU_{\S}(\Upsilon) = [IL_{\S}(\Upsilon^c)]^c$.

(3) The increasing and decreasing boundary approximations of Υ are, respectively, defined by $IB_{\S}(\Upsilon) = IU_{\S}(\Upsilon) \setminus IL_{\S}(\Upsilon)$ and $DB_{\S}(\Upsilon) = DU_{\S}(\Upsilon) \setminus DL_{\S}(\Upsilon)$.

The concept of nano topological space was initially introduced by Thivagar and Richard [18]. This topological space contains the lower approximation, the upper approximation and the boundary, which are constructed using an equivalence relation. El Bably and Abo Tabl [10] extended the concept of nano topology by replacing the equivalence relation with an arbitrary relation for approximation spaces which satisfies a special condition.

Theorem 2.12. [10] Let Σ be a finite set and let $\S$ be any binary relation. $\mathcal{L}_*(\Upsilon)$, $\mathcal{U}^*(\Upsilon)$ and $\mathcal{B}(\Upsilon)$ are the generalized lower approximation, the generalized upper approximation, and the boundary of a subset $\Upsilon \subseteq \Sigma$, respectively. The class $\tau_{GN} = \{\Sigma, \emptyset, \mathcal{L}_*(\Upsilon), \mathcal{U}^*(\Upsilon), \mathcal{B}(\Upsilon)\}$ forms a topology on Σ whenever $\mathcal{L}_*(\Upsilon) \subseteq \Upsilon \subseteq \mathcal{U}^*(\Upsilon)$ for every $\Upsilon \subseteq \Sigma$ and τ_{GN} is called a generalized nano topology (briefly, GN topology) on Σ associated with $\Upsilon \subseteq \Sigma$.

Corollary 2.13. [10] The basis of the GN topology is given by the class $\mathfrak{B}_{GN} = \{\Sigma, \mathcal{L}_*(\Upsilon), \mathcal{B}(\Upsilon)\}$.

3. The Increasing and Decreasing j -Neighborhood Classes

In this section, we present the concepts of the increasing and decreasing j -neighborhood classes using j -neighborhoods and partial order relations. We analyze the fundamental characteristics of these classes. Furthermore, we establish the relationships between these classes, both j -neighborhood classes and increasing and decreasing equivalence classes, under certain conditions. We provide some examples and counterexamples to support the findings.

Definition 3.1. Let $(\Sigma, \S, \varphi_j)$ be a j - $\mathcal{NS}$ and $\preceq$ be a partial order relation. The increasing j -neighborhood classes of $x \in \Sigma$ are denoted by $\mathcal{IN}_j(x)$ and are defined for each $j \in J = \{r, l, i, u\}$ as follows, where $i_{\preceq}(y) = \{z \in \Sigma : y \preceq z\}$ denotes the increasing set of $y \in \Sigma$.

$$\mathcal{IN}_r(x) = \cup\{i_{\preceq}(y) : y \in \Sigma, y \in \mathcal{N}_r(x)\}$$

$$\mathcal{IN}_l(x) = \cup\{i_{\preceq}(y) : y \in \Sigma, y \in \mathcal{N}_l(x)\}$$

$$\mathcal{IN}_i(x) = \mathcal{IN}_r(x) \cap \mathcal{IN}_l(x)$$

$$\mathcal{IN}_u(x) = \mathcal{IN}_r(x) \cup \mathcal{IN}_l(x)$$

Here, the concept of an increasing j -neighborhood class refers to the increasing right, left, intersection, and union neighborhood classes for $j \in J$, respectively.

Definition 3.2. Let $(\Sigma, \S, \varphi_j)$ be a j - $\mathcal{NS}$ and $\preceq$ be a partial order relation. The decreasing j -neighborhood classes of $x \in \Sigma$ are denoted by $\mathcal{DN}_j(x)$ and are defined for each $j \in J$ as follows, where $d_{\preceq}(y) = \{z \in \Sigma : z \preceq y\}$ denotes the decreasing set of $y \in \Sigma$.

$$\mathcal{DN}_r(x) = \cup\{d_{\preceq}(y) : y \in \Sigma, y \in \mathcal{N}_r(x)\}$$

$$\mathcal{DN}_l(x) = \cup\{d_{\preceq}(y) : y \in \Sigma, y \in \mathcal{N}_l(x)\}$$

$$\mathcal{DN}_i(x) = \mathcal{DN}_r(x) \cap \mathcal{DN}_l(x)$$

$$\mathcal{DN}_u(x) = \mathcal{DN}_r(x) \cup \mathcal{DN}_l(x)$$

Here, the concept of decreasing j -neighborhood class refer to decreasing right, left, intersection, union neighborhood classes for $j \in J$, respectively.

Definition 3.3. The increasing and decreasing j -neighborhood classes are referred to as the ordered j -neighborhood classes.

The following remark shows that, for certain binary relations, the ordered j -neighborhood classes are equivalent to the j -neighborhood classes as well as to the increasing and decreasing equivalence classes. Therefore, it is clear that the ordered j -neighborhood classes constitute a generalization of both j -neighborhoods given in [1, 19, 20], and the increasing and decreasing equivalence classes defined by Shalil et al. [17].

Remark 3.4.

1. If $\S$ is an equivalence relation, then Definition 3.1 and Definition 3.2 coincide with Definition 2.10.

2. If $\S$ is any binary relation and $\preceq$ is an equality relation on Σ , then the increasing and decreasing j -neighborhood classes are equivalent for each $j \in J$. In addition, Definition 3.1 and Definition 3.2 coincide with Definition 2.3 for $j = \{r\}$ and Definition 2.6 for each $j \in \{l, i, u\}$.

Proposition 3.5. Let $(\Sigma, \S, \varphi_j)$ be a j - $\mathcal{NS}$ and $\preceq$ be a partial order relation. The following properties hold for all $x, y \in \Sigma$:

- (1) $\mathcal{N}_j(x) \subseteq \mathcal{I}\mathcal{N}_j(x)$ and $\mathcal{N}_j(x) \subseteq \mathcal{D}\mathcal{N}_j(x)$ for each $j \in J$.
- (2) (a) $\mathcal{I}\mathcal{N}_i(x) \subseteq \mathcal{I}\mathcal{N}_r(x) \subseteq \mathcal{I}\mathcal{N}_u(x)$ and $\mathcal{I}\mathcal{N}_i(x) \subseteq \mathcal{I}\mathcal{N}_l(x) \subseteq \mathcal{I}\mathcal{N}_u(x)$.
(b) $\mathcal{D}\mathcal{N}_i(x) \subseteq \mathcal{D}\mathcal{N}_r(x) \subseteq \mathcal{D}\mathcal{N}_u(x)$ and $\mathcal{D}\mathcal{N}_i(x) \subseteq \mathcal{D}\mathcal{N}_l(x) \subseteq \mathcal{D}\mathcal{N}_u(x)$.
- (3) If $\mathcal{N}_j(x) \subseteq \mathcal{N}_j(y)$ then $\mathcal{I}\mathcal{N}_j(x) \subseteq \mathcal{I}\mathcal{N}_j(y)$ and $\mathcal{D}\mathcal{N}_j(x) \subseteq \mathcal{D}\mathcal{N}_j(y)$ for each $j \in \{r, l, u\}$.

Proof.

- (1) Let $y \in \mathcal{N}_r(x)$. Since $\preceq$ is a reflexive relation, we have $y \in \cup\{i_{\preceq}(y) : y \in \Sigma, y \in \mathcal{N}_r(x)\}$. Hence, we obtain $\mathcal{N}_r(x) \subseteq \mathcal{I}\mathcal{N}_r(x)$. Similarly, it follows that $\mathcal{N}_l(x) \subseteq \mathcal{I}\mathcal{N}_l(x)$. This implies that $\mathcal{N}_i(x) = \mathcal{N}_r(x) \cap \mathcal{N}_l(x) \subseteq \mathcal{I}\mathcal{N}_r(x) \cap \mathcal{I}\mathcal{N}_l(x) = \mathcal{I}\mathcal{N}_i(x)$ and $\mathcal{N}_u(x) = \mathcal{N}_r(x) \cup \mathcal{N}_l(x) \subseteq \mathcal{I}\mathcal{N}_r(x) \cup \mathcal{I}\mathcal{N}_l(x) = \mathcal{I}\mathcal{N}_u(x)$. Analogously, it can be shown that $\mathcal{N}_j(x) \subseteq \mathcal{D}\mathcal{N}_j(x)$ for all $x \in \Sigma$ and for each $j \in J$.
- (2) Their proofs are straightforward from the definitions of the ordered j -neighborhood classes.
- (3) Let $w \in \mathcal{I}\mathcal{N}_r(x) = \cup\{i_{\preceq}(z) : z \in \Sigma, z \in \mathcal{N}_r(x)\}$. Since $\mathcal{N}_r(x) \subseteq \mathcal{N}_r(y)$, we have $w \in \mathcal{I}\mathcal{N}_r(y) = \{i_{\preceq}(z) : z \in \Sigma, z \in \mathcal{N}_r(y)\}$. Therefore, we obtain $\mathcal{I}\mathcal{N}_r(x) \subseteq \mathcal{I}\mathcal{N}_r(y)$. In a similar manner, we derive $\mathcal{I}\mathcal{N}_l(x) \subseteq \mathcal{I}\mathcal{N}_l(y)$. In addition, let $w \in \mathcal{I}\mathcal{N}_u(x)$. Then, we have $w \in \mathcal{I}\mathcal{N}_r(x)$ or $w \in \mathcal{I}\mathcal{N}_l(x)$. Assume that $w \in \mathcal{I}\mathcal{N}_r(x) = \cup\{i_{\preceq}(z) : z \in \Sigma, z \in \mathcal{N}_r(x)\}$. Since $\mathcal{N}_r(x) \subseteq \mathcal{N}_u(x) \subseteq \mathcal{N}_u(y) = \mathcal{N}_r(y) \cup \mathcal{N}_l(y)$ and by Proposition 3.5 (1), we obtain $w \in \mathcal{I}\mathcal{N}_r(y)$ or $w \in \mathcal{I}\mathcal{N}_l(y)$. Thus, $w \in \mathcal{I}\mathcal{N}_u(y)$. On the other hand, let $w \in \mathcal{I}\mathcal{N}_l(x) = \cup\{i_{\preceq}(z) : z \in \Sigma, z \in \mathcal{N}_l(x)\}$. Since $\mathcal{N}_l(x) \subseteq \mathcal{N}_u(x) \subseteq \mathcal{N}_u(y) = \mathcal{N}_r(y) \cup \mathcal{N}_l(y)$ and from Proposition 3.5 (1), we obtain $w \in \mathcal{I}\mathcal{N}_r(y)$ or $w \in \mathcal{I}\mathcal{N}_l(y)$. Hence, $w \in \mathcal{I}\mathcal{N}_u(y)$. As a result, $\mathcal{I}\mathcal{N}_u(x) \subseteq \mathcal{I}\mathcal{N}_u(y)$. Analogously, it can be shown that $\mathcal{D}\mathcal{N}_j(x) \subseteq \mathcal{D}\mathcal{N}_j(y)$. $\square$

The following example illustrates that, in general, Proposition 3.5 (3) is not valid for $j = i$.

Example 3.6. Let $\preceq = \Delta \cup \{(\rho_1, \rho_3), (\rho_3, \rho_4), (\rho_1, \rho_4)\}$ be a partial order relation, and $\S = \{(\rho_1, \rho_2), (\rho_2, \rho_3), (\rho_3, \rho_4), (\rho_4, \rho_2)\}$ be a relation on $\Sigma = \{\rho_1, \rho_2, \rho_3, \rho_4\}$. Then, we obtain $\mathcal{N}_i(\rho_2) = \emptyset \subseteq \emptyset = \mathcal{N}_i(\rho_1)$ but $\mathcal{I}\mathcal{N}_i(\rho_2) = \{\rho_3, \rho_4\} \not\subseteq \emptyset = \mathcal{I}\mathcal{N}_i(\rho_1)$ and $\mathcal{D}\mathcal{N}_i(\rho_2) = \{\rho_1, \rho_3\} \not\subseteq \emptyset = \mathcal{D}\mathcal{N}_i(\rho_1)$.

Proposition 3.7. Let $(\Sigma, \S, \varphi_j)$ be a j - $\mathcal{NS}$ and $\preceq$ be a partial order relation. If $\S$ is an equality relation, then the following properties hold for all $x \in \Sigma$ and for each $j \in J$:

- (1) $\mathcal{I}\mathcal{N}_j(x) = i_{\preceq}(x)$ and $\mathcal{D}\mathcal{N}_j(x) = d_{\preceq}(x)$.
- (2) (a) If $y \in \mathcal{I}\mathcal{N}_j(x)$ then $\mathcal{I}\mathcal{N}_j(y) \subseteq \mathcal{I}\mathcal{N}_j(x)$.
(b) If $y \in \mathcal{D}\mathcal{N}_j(x)$ then $\mathcal{D}\mathcal{N}_j(y) \subseteq \mathcal{D}\mathcal{N}_j(x)$.

Proof.

- (1) If $\S$ is an equality relation, then $\mathcal{N}_j(x) = \{x\}$ for all $x \in \Sigma$ and for each $j \in J$. It follows that $\mathcal{I}\mathcal{N}_j(x) = i_{\preceq}(x)$ by Definition 3.1 and $\mathcal{D}\mathcal{N}_j(x) = d_{\preceq}(x)$ by Definition 3.2.
- (2) Let $z \in \mathcal{I}\mathcal{N}_j(y)$. From (1), we have $z \in i_{\preceq}(y)$. Since $y \in \mathcal{I}\mathcal{N}_j(x)$, we get $y \in i_{\preceq}(x)$. Obviously, these require that $z \in i_{\preceq}(x)$ by transitivity. So, $z \in \mathcal{I}\mathcal{N}_j(x)$. The other proof is carried out in the same way. $\square$

Proposition 3.7 (2) does not generally hold when $\S$ is not an equality relation, as shown in the following example.

Example 3.8. Let $\preceq = \Delta \cup \{(\rho_1, \rho_3), (\rho_3, \rho_4), (\rho_1, \rho_4)\}$ be a partial order relation and $\S = \Delta \cup \{(\rho_2, \rho_1), (\rho_4, \rho_2)\}$ be a relation on $\Sigma = \{\rho_1, \rho_2, \rho_3, \rho_4\}$. Then, we get $\rho_4 \in \mathcal{I}\mathcal{N}_r(\rho_1) = \{\rho_1, \rho_3, \rho_4\}$ but $\mathcal{I}\mathcal{N}_r(\rho_4) = \{\rho_2, \rho_4\} \not\subseteq \{\rho_1, \rho_3, \rho_4\} = \mathcal{I}\mathcal{N}_r(\rho_1)$.

Proposition 3.9. Let $(\Sigma, \S, \varphi_j)$ be a j - $\mathcal{NS}$ and $\preceq$ be a partial order relation. If $\S$ is a reflexive relation, then the following properties hold for all $x \in \Sigma$ and for each $j \in J$:

- (1) $x \in \mathcal{I}\mathcal{N}_j(x)$ and $x \in \mathcal{D}\mathcal{N}_j(x)$.
- (2) $\mathcal{I}\mathcal{N}_j(x) \neq \emptyset$ and $\mathcal{D}\mathcal{N}_j(x) \neq \emptyset$.

Proof.

- (1) Suppose that $\S$ is a reflexive relation. Then $x \in \mathcal{N}_j(x)$ for each $j \in J$. Moreover, since $\preceq$ is also a reflexive relation, we obtain $x \in \mathcal{I}\mathcal{N}_j(x)$ and $x \in \mathcal{D}\mathcal{N}_j(x)$.
- (2) They follow immediately from (1). $\square$

Proposition 3.9 does not hold in general when $\S$ is not a reflexive relation, as illustrated by the following example.

Example 3.10. Let $\preceq = \Delta \cup \{(\rho_1, \rho_3), (\rho_3, \rho_4), (\rho_1, \rho_4)\}$ be a partial order relation and let $\S = \{(\rho_1, \rho_2), (\rho_2, \rho_3), (\rho_1, \rho_3), (\rho_4, \rho_1), (\rho_4, \rho_2), (\rho_4, \rho_3)\}$ be a relation on $\Sigma = \{\rho_1, \rho_2, \rho_3, \rho_4\}$. Then, we obtain that $\rho_2 \notin \mathcal{I}\mathcal{N}_r(\rho_2) = \{\rho_3, \rho_4\}$ and $\rho_2 \notin \mathcal{D}\mathcal{N}_r(\rho_2) = \{\rho_1, \rho_3\}$.

Proposition 3.11. Let $(\Sigma, \S, \varphi_j)$ be a j - $\mathcal{NS}$ and $\preceq$ be a partial order relation. If $\S$ is a symmetric relation, then the following properties hold for all $x \in \Sigma$:

- (1) $\mathcal{I}\mathcal{N}_i(x) = \mathcal{I}\mathcal{N}_r(x) = \mathcal{I}\mathcal{N}_l(x) = \mathcal{I}\mathcal{N}_u(x)$.
- (2) $\mathcal{D}\mathcal{N}_i(x) = \mathcal{D}\mathcal{N}_r(x) = \mathcal{D}\mathcal{N}_l(x) = \mathcal{D}\mathcal{N}_u(x)$.

Proof. The proofs are straightforward, since $\mathcal{N}_r(x) = \mathcal{N}_l(x)$ whenever $\S$ is a symmetric relation. $\square$

Proposition 3.12. Let $(\Sigma, \S, \varphi_j)$ be a j - $\mathcal{NS}$ and $\preceq$ be a partial order relation. If $\S$ is a serial relation, then $\mathcal{I}\mathcal{N}_r(x) \neq \emptyset$ and $\mathcal{D}\mathcal{N}_r(x) \neq \emptyset$ for all $x \in \Sigma$.

Proof. Let $\S$ be a serial relation. Then $\mathcal{N}_r(x) \neq \emptyset$ for all $x \in \Sigma$. Consequently, since $\preceq$ is a reflexive relation, we have $\mathcal{I}\mathcal{N}_r(x) \neq \emptyset$ and $\mathcal{D}\mathcal{N}_r(x) \neq \emptyset$ for all $x \in \Sigma$. $\square$

Proposition 3.12 does not hold in general when $\S$ is not a serial relation, as illustrated by the following example.

Example 3.13. Consider Example 3.10, we have $\mathcal{I}\mathcal{N}_r(\rho_3) = \emptyset$ and $\mathcal{D}\mathcal{N}_r(\rho_3) = \emptyset$.

Proposition 3.14. Let $(\Sigma, \S, \varphi_j)$ be a j - $\mathcal{NS}$ and $\preceq$ be a partial order relation. If $\S$ is an inverse serial relation, then $\mathcal{I}\mathcal{N}_l(x) \neq \emptyset$ and $\mathcal{D}\mathcal{N}_l(x) \neq \emptyset$ for all $x \in \Sigma$.

Proof. Let $\S$ be an inverse serial relation. Then $\mathcal{N}_l(x) \neq \emptyset$ for all $x \in \Sigma$. It follows that $\mathcal{I}\mathcal{N}_l(x) \neq \emptyset$ and $\mathcal{D}\mathcal{N}_l(x) \neq \emptyset$ for all $x \in \Sigma$. $\square$

Proposition 3.14 does not hold in general when $\S$ is not inverse serial, as shown in the following example.

Example 3.15. Consider Example 3.10, we have $\mathcal{I}\mathcal{N}_l(\rho_4) = \emptyset$ and $\mathcal{D}\mathcal{N}_l(\rho_4) = \emptyset$.

4. Ordered j -Approximation Spaces Based on Ordered j -Neighborhood Classes

In this section, we use the ordered j -neighborhood classes to extend the ordered approximation spaces. First, we define the lower and upper approximations based on these classes and investigate their fundamental properties. Furthermore, we compare these novel approximations with the prior ones, i.e., [9, 17, 20], for specific binary relations. Then, we provide numerical examples to illustrate the effectiveness of our approaches. Finally, we study the GN topologies induced by these approaches.

Definition 4.1. The ordered j -approximation space (briefly, j - $\mathcal{OAS}$) is denoted by $(\Sigma, \mathcal{N}_j, \preceq)$, where $(\Sigma, \S, \varphi_j)$ is a j - $\mathcal{NS}$ and $\preceq$ is a partial order relation defined on Σ .

Definition 4.2. Let $(\Sigma, \mathcal{N}_j, \preceq)$ be a j - $\mathcal{OAS}$. The following notions are defined for every $\Upsilon \subseteq \Sigma$:

1. The increasing and decreasing j -lower approximations of Υ based on ordered j -neighborhood classes are defined, respectively, as follows.

$$\mathcal{I}\mathcal{L}_j(\Upsilon) = \cup\{\mathcal{I}\mathcal{N}_j(x) : \mathcal{N}_j(x) \subseteq \Upsilon \cap \Upsilon\}$$

$$\mathcal{D}\mathcal{L}_j(\Upsilon) = \cup\{\mathcal{D}\mathcal{N}_j(x) : \mathcal{N}_j(x) \subseteq \Upsilon \cap \Upsilon\}$$

2. The increasing and decreasing j -upper approximations of Υ based on ordered j -neighborhood classes are defined, respectively, as follows.

$$\mathcal{I}\mathcal{U}_j(\Upsilon) = [\mathcal{D}\mathcal{L}_j(\Upsilon^c)]^c \text{ and } \mathcal{D}\mathcal{U}_j(\Upsilon) = [\mathcal{I}\mathcal{L}_j(\Upsilon^c)]^c$$

3. The increasing and decreasing j -boundaries of Υ are defined by $\mathcal{I}\mathcal{B}_j(\Upsilon) = \mathcal{I}\mathcal{U}_j(\Upsilon) \setminus \mathcal{I}\mathcal{L}_j(\Upsilon)$ and $\mathcal{D}\mathcal{B}_j(\Upsilon) = \mathcal{D}\mathcal{U}_j(\Upsilon) \setminus \mathcal{D}\mathcal{L}_j(\Upsilon)$, respectively.

4. The increasing and decreasing j -accuracies of Υ are, respectively, defined by $i_{\preceq}\alpha_j(\Upsilon) = \frac{|\mathcal{I}\mathcal{L}_j(\Upsilon)|}{|\mathcal{I}\mathcal{U}_j(\Upsilon)|}$ and $d_{\preceq}\alpha_j(\Upsilon) = \frac{|\mathcal{D}\mathcal{L}_j(\Upsilon)|}{|\mathcal{D}\mathcal{U}_j(\Upsilon)|}$ where $|\mathcal{I}\mathcal{U}_j(\Upsilon)| \neq 0$ and $|\mathcal{D}\mathcal{U}_j(\Upsilon)| \neq 0$.

It is clear that $i_{\preceq}\alpha_j(\Upsilon) \in [0, 1]$ and $d_{\preceq}\alpha_j(\Upsilon) \in [0, 1]$. If $\mathcal{I}\mathcal{L}_j(\Upsilon) = \mathcal{I}\mathcal{U}_j(\Upsilon)$, then Υ is called the increasing j -exact set, and if $\mathcal{D}\mathcal{L}_j(\Upsilon) = \mathcal{D}\mathcal{U}_j(\Upsilon)$, then Υ is called the decreasing j -exact set. If $\mathcal{I}\mathcal{L}_j(\Upsilon) \neq \mathcal{I}\mathcal{U}_j(\Upsilon)$, then Υ is called the increasing j -rough set, and if $\mathcal{D}\mathcal{L}_j(\Upsilon) \neq \mathcal{D}\mathcal{U}_j(\Upsilon)$, then Υ is called the decreasing j -rough set.

Proposition 4.3. Let $(\Sigma, \mathcal{N}_j, \preceq)$ be a j - $\mathcal{OAS}$. For all $\Upsilon, \Gamma \subseteq \Sigma$, the following properties hold:

- | | |
|---|---|
| (IJ1) $\mathcal{I}\mathcal{L}_j(\Upsilon) \subseteq \Upsilon \subseteq \mathcal{I}\mathcal{U}_j(\Upsilon)$ | (DJ1) $\mathcal{D}\mathcal{L}_j(\Upsilon) \subseteq \Upsilon \subseteq \mathcal{D}\mathcal{U}_j(\Upsilon)$ |
| (IJ2) $\mathcal{I}\mathcal{L}_j(\emptyset) = \emptyset$ and $\mathcal{I}\mathcal{U}_j(\Sigma) = \Sigma$ | (DJ2) $\mathcal{D}\mathcal{L}_j(\emptyset) = \emptyset$ and $\mathcal{D}\mathcal{U}_j(\Sigma) = \Sigma$ |
| (IJ3) If $\Gamma \subseteq \Upsilon$ then $\mathcal{I}\mathcal{L}_j(\Gamma) \subseteq \mathcal{I}\mathcal{L}_j(\Upsilon)$ | (DJ3) If $\Gamma \subseteq \Upsilon$ then $\mathcal{D}\mathcal{L}_j(\Gamma) \subseteq \mathcal{D}\mathcal{L}_j(\Upsilon)$ |
| (IJ4) If $\Gamma \subseteq \Upsilon$ then $\mathcal{I}\mathcal{U}_j(\Gamma) \subseteq \mathcal{I}\mathcal{U}_j(\Upsilon)$ | (DJ4) If $\Gamma \subseteq \Upsilon$ then $\mathcal{D}\mathcal{U}_j(\Gamma) \subseteq \mathcal{D}\mathcal{U}_j(\Upsilon)$ |
| (IJ5) $\mathcal{I}\mathcal{L}_j(\Gamma \cap \Upsilon) \subseteq \mathcal{I}\mathcal{L}_j(\Gamma) \cap \mathcal{I}\mathcal{L}_j(\Upsilon)$ | (DJ5) $\mathcal{D}\mathcal{L}_j(\Gamma \cap \Upsilon) \subseteq \mathcal{D}\mathcal{L}_j(\Gamma) \cap \mathcal{D}\mathcal{L}_j(\Upsilon)$ |
| (IJ6) $\mathcal{I}\mathcal{L}_j(\Gamma) \cup \mathcal{I}\mathcal{L}_j(\Upsilon) \subseteq \mathcal{I}\mathcal{L}_j(\Gamma \cup \Upsilon)$ | (DJ6) $\mathcal{D}\mathcal{L}_j(\Gamma) \cup \mathcal{D}\mathcal{L}_j(\Upsilon) \subseteq \mathcal{D}\mathcal{L}_j(\Gamma \cup \Upsilon)$ |
| (IJ7) $\mathcal{I}\mathcal{U}_j(\Gamma \cap \Upsilon) \subseteq \mathcal{I}\mathcal{U}_j(\Gamma) \cap \mathcal{I}\mathcal{U}_j(\Upsilon)$ | (DJ7) $\mathcal{D}\mathcal{U}_j(\Gamma \cap \Upsilon) \subseteq \mathcal{D}\mathcal{U}_j(\Gamma) \cap \mathcal{D}\mathcal{U}_j(\Upsilon)$ |
| (IJ8) $\mathcal{I}\mathcal{U}_j(\Gamma) \cup \mathcal{I}\mathcal{U}_j(\Upsilon) \subseteq \mathcal{I}\mathcal{U}_j(\Gamma \cup \Upsilon)$ | (DJ8) $\mathcal{D}\mathcal{U}_j(\Gamma) \cup \mathcal{D}\mathcal{U}_j(\Upsilon) \subseteq \mathcal{D}\mathcal{U}_j(\Gamma \cup \Upsilon)$ |
| (IJ9) $\mathcal{I}\mathcal{L}_j(\mathcal{I}\mathcal{L}_j(\Upsilon)) = \mathcal{I}\mathcal{L}_j(\Upsilon)$ | (DJ9) $\mathcal{D}\mathcal{L}_j(\mathcal{D}\mathcal{L}_j(\Upsilon)) = \mathcal{D}\mathcal{L}_j(\Upsilon)$ |
| (IJ10) $\mathcal{I}\mathcal{U}_j(\mathcal{I}\mathcal{U}_j(\Upsilon)) = \mathcal{I}\mathcal{U}_j(\Upsilon)$ | (DJ10) $\mathcal{D}\mathcal{U}_j(\mathcal{D}\mathcal{U}_j(\Upsilon)) = \mathcal{D}\mathcal{U}_j(\Upsilon)$ |
| (IJ11) $\mathcal{I}\mathcal{L}_j(\mathcal{I}\mathcal{U}_j(\Upsilon)) \subseteq \mathcal{I}\mathcal{U}_j(\Upsilon)$ | (DJ11) $\mathcal{D}\mathcal{L}_j(\mathcal{D}\mathcal{U}_j(\Upsilon)) \subseteq \mathcal{D}\mathcal{U}_j(\Upsilon)$ |
| (IJ12) $\mathcal{I}\mathcal{L}_j(\Upsilon) \subseteq \mathcal{I}\mathcal{U}_j(\mathcal{I}\mathcal{L}_j(\Upsilon))$ | (DJ12) $\mathcal{D}\mathcal{L}_j(\Upsilon) \subseteq \mathcal{D}\mathcal{U}_j(\mathcal{D}\mathcal{L}_j(\Upsilon))$ |
| (IJ13) $[\mathcal{I}\mathcal{L}_j(\Upsilon)]^c = \mathcal{D}\mathcal{U}_j(\Upsilon^c)$ | (DJ13) $[\mathcal{D}\mathcal{L}_j(\Upsilon)]^c = \mathcal{I}\mathcal{U}_j(\Upsilon^c)$ |

Proof.

(IJ1) & (DJ1) From Definition 4.2, we have $\mathcal{I}\mathcal{L}_j(\Upsilon) \subseteq \Upsilon$ and $\mathcal{D}\mathcal{L}_j(\Upsilon) \subseteq \Upsilon$. This implies $\mathcal{I}\mathcal{L}_j(\Upsilon^c) \subseteq \Upsilon^c$ and $\mathcal{D}\mathcal{L}_j(\Upsilon^c) \subseteq \Upsilon^c$. Then, we obtain that $\mathcal{D}\mathcal{U}_j(\Upsilon) = [\mathcal{I}\mathcal{L}_j(\Upsilon^c)]^c \supseteq [\Upsilon^c]^c = \Upsilon$ and $\mathcal{I}\mathcal{U}_j(\Upsilon) = [\mathcal{D}\mathcal{L}_j(\Upsilon^c)]^c \supseteq [\Upsilon^c]^c = \Upsilon$.

(IJ2) & (DJ2) Since $\mathcal{J}\mathcal{L}_j(\Upsilon) \subseteq \Upsilon$ and $\mathcal{D}\mathcal{L}_j(\Upsilon) \subseteq \Upsilon$, we have $\mathcal{J}\mathcal{L}_j(\emptyset) = \emptyset$ and $\mathcal{D}\mathcal{L}_j(\emptyset) = \emptyset$. Then, we obtain $\mathcal{D}\mathcal{U}_j(\Sigma) = [\mathcal{J}\mathcal{L}_j(\Sigma^c)]^c = [\mathcal{J}\mathcal{L}_j(\emptyset)]^c = [\emptyset]^c = \Sigma$ and $\mathcal{J}\mathcal{U}_j(\Sigma) = [\mathcal{D}\mathcal{L}_j(\Sigma^c)]^c = [\mathcal{D}\mathcal{L}_j(\emptyset)]^c = [\emptyset]^c = \Sigma$.

(IJ3) & (DJ3) Let $\Gamma \subseteq \Upsilon$. Then $\mathcal{J}\mathcal{L}_j(\Gamma) = \cup\{\mathcal{J}\mathcal{N}_j(x) : \mathcal{N}_j(x) \subseteq \Gamma\} \cap \Gamma \subseteq \cup\{\mathcal{J}\mathcal{N}_j(x) : \mathcal{N}_j(x) \subseteq \Gamma\} \cap \Upsilon \subseteq \cup\{\mathcal{J}\mathcal{N}_j(x) : \mathcal{N}_j(x) \subseteq \Upsilon\} \cap \Upsilon = \mathcal{J}\mathcal{L}_j(\Upsilon)$ and similarly, $\mathcal{D}\mathcal{L}_j(\Gamma) \subseteq \mathcal{D}\mathcal{L}_j(\Upsilon)$ are obtained.

(IJ4) & (DJ4) Let $\Gamma \subseteq \Upsilon$. Then we have $\Upsilon^c \subseteq \Gamma^c$. This implies that $\mathcal{J}\mathcal{L}_j(\Upsilon^c) \subseteq \mathcal{J}\mathcal{L}_j(\Gamma^c)$ and $\mathcal{D}\mathcal{L}_j(\Upsilon^c) \subseteq \mathcal{D}\mathcal{L}_j(\Gamma^c)$ by (IJ3) and (DJ3). Then, we obtain $\mathcal{D}\mathcal{U}_j(\Gamma) = [\mathcal{J}\mathcal{L}_j(\Gamma^c)]^c \subseteq [\mathcal{J}\mathcal{L}_j(\Upsilon^c)]^c = \mathcal{D}\mathcal{U}_j(\Upsilon)$ and $\mathcal{J}\mathcal{U}_j(\Gamma) = [\mathcal{D}\mathcal{L}_j(\Gamma^c)]^c \subseteq [\mathcal{D}\mathcal{L}_j(\Upsilon^c)]^c = \mathcal{J}\mathcal{U}_j(\Upsilon)$.

(IJ9) & (DJ9) From (IJ1) and (IJ3), we have $\mathcal{J}\mathcal{L}_j(\mathcal{J}\mathcal{L}_j(\Upsilon)) \subseteq \mathcal{J}\mathcal{L}_j(\Upsilon)$. Let $y \in \mathcal{J}\mathcal{L}_j(\Upsilon) = \cup\{\mathcal{J}\mathcal{N}_j(x) : \mathcal{N}_j(x) \subseteq \Upsilon\} \cap \Upsilon$. This implies $y \in \Upsilon$ and there exists $\mathcal{N}_j(x) \subseteq \Upsilon$ such that $y \in \mathcal{J}\mathcal{N}_j(x)$. From Proposition 3.5 (1), we have $\mathcal{N}_j(x) \subseteq \mathcal{J}\mathcal{N}_j(x)$. Then, we derive that $\mathcal{N}_j(x) \subseteq \mathcal{J}\mathcal{N}_j(x) \cap \Upsilon \subseteq \cup\{\mathcal{J}\mathcal{N}_j(x) : \mathcal{N}_j(x) \subseteq \Upsilon\} \cap \Upsilon = \mathcal{J}\mathcal{L}_j(\Upsilon)$. Consequently, $y \in \mathcal{J}\mathcal{L}_j(\Upsilon)$ and there exists $\mathcal{N}_j(x) \subseteq \mathcal{J}\mathcal{L}_j(\Upsilon)$ such that $y \in \mathcal{J}\mathcal{N}_j(x)$, that is, $y \in \mathcal{J}\mathcal{L}_j(\mathcal{J}\mathcal{L}_j(\Upsilon))$. So $\mathcal{J}\mathcal{L}_j(\mathcal{J}\mathcal{L}_j(\Upsilon)) = \mathcal{J}\mathcal{L}_j(\Upsilon)$. Similarly, $\mathcal{D}\mathcal{L}_j(\mathcal{D}\mathcal{L}_j(\Upsilon)) = \mathcal{D}\mathcal{L}_j(\Upsilon)$ is obtained.

The proofs of (IJ5)-(IJ8), (IJ11), (IJ12) and (DJ5)-(DJ8), (DJ11), (DJ12) are clear from (IJ3) and (DJ3). (IJ10) and (DJ10) are obtained from (IJ4), (IJ9), (DJ4) and (DJ9). (IJ13) and (DJ13) are obvious by Definition 4.2. $\square$

Table 1 summarizes the comparison between the properties of Pawlak approximation spaces and the second type of approximations given by Atef et al. (see [9, 12]) and Yao (see [2, 20]), and the ordered j -approximations.

Table 1: Comparison between the properties of approximations

Pawlak approximation	The second type of approximations given by Atef et al. and Yao, and the ordered j -approximations
$\underline{R}(\Upsilon) = [\overline{R}(\Upsilon^c)]^c$	$\checkmark$
$\underline{R}(\Sigma) = \Sigma$	
$\underline{R}(\Upsilon \cap \Gamma) = \underline{R}(\Upsilon) \cap \underline{R}(\Gamma)$	
$\underline{R}(\Upsilon \cup \Gamma) \supseteq \underline{R}(\Upsilon) \cup \underline{R}(\Gamma)$	$\checkmark$
If $\Upsilon \subseteq \Gamma$ then $\underline{R}(\Upsilon) \subseteq \underline{R}(\Gamma)$	$\checkmark$
$\underline{R}(\emptyset) = \emptyset$	$\checkmark$
$\underline{R}(\Upsilon) \subseteq \Upsilon$	$\checkmark$
$\underline{R}(\underline{R}(\Upsilon)) = \underline{R}(\Upsilon)$	$\checkmark$
$\overline{R}(\Upsilon) = [\underline{R}(\Upsilon^c)]^c$	$\checkmark$
$\overline{R}(\emptyset) = \emptyset$	
$\overline{R}(\Upsilon \cup \Gamma) = \overline{R}(\Upsilon) \cup \overline{R}(\Gamma)$	
$\overline{R}(\Upsilon \cap \Gamma) \subseteq \overline{R}(\Upsilon) \cap \overline{R}(\Gamma)$	$\checkmark$
If $\Upsilon \subseteq \Gamma$ then $\overline{R}(\Upsilon) \subseteq \overline{R}(\Gamma)$	$\checkmark$
$\overline{R}(\Sigma) = \Sigma$	$\checkmark$
$\Upsilon \subseteq \overline{R}(\Upsilon)$	$\checkmark$
$\overline{R}(\overline{R}(\Upsilon)) = \overline{R}(\Upsilon)$	$\checkmark$

The following examples confirm that the equalities in (IJ1), (IJ5)-(IJ8), (IJ11)-(IJ12) (and similarly (DJ1), (DJ5)-(DJ8), (DJ11)-(DJ12)) of Proposition 4.3 do not hold in general.

Example 4.4. Let $\preceq = \Delta \cup \{(\rho_1, \rho_3), (\rho_3, \rho_4), (\rho_1, \rho_4)\}$ be a partial order relation and let $\S = \{(\rho_1, \rho_3), (\rho_3, \rho_1), (\rho_2, \rho_3), (\rho_3, \rho_2), (\rho_1, \rho_1)\}$ be a relation on $\Sigma = \{\rho_1, \rho_2, \rho_3, \rho_4\}$. We get $\mathcal{D}\mathcal{L}_r(\Sigma) = \{\rho_1, \rho_2, \rho_3\} \neq \Sigma$ and $\mathcal{J}\mathcal{U}_r(\emptyset) = \{\rho_4\} \neq \emptyset$.

Let $\mathcal{A} = \{\rho_1\}$ and $\mathcal{B} = \{\rho_2\}$. We have $\mathcal{J}\mathcal{L}_r(\mathcal{A} \cup \mathcal{B}) = \{\rho_1, \rho_2\} \neq \emptyset = \mathcal{J}\mathcal{L}_r(\mathcal{A}) \cup \mathcal{J}\mathcal{L}_r(\mathcal{B})$ and $\mathcal{D}\mathcal{L}_r(\mathcal{A} \cup \mathcal{B}) = \{\rho_1, \rho_2\} \neq \emptyset = \mathcal{D}\mathcal{L}_r(\mathcal{A}) \cup \mathcal{D}\mathcal{L}_r(\mathcal{B})$.

Moreover, $\mathcal{J}\mathcal{U}_r(\mathcal{A} \cap \mathcal{B}) = \{\rho_4\} \neq \{\rho_2, \rho_4\} = \mathcal{J}\mathcal{U}_r(\mathcal{A}) \cap \mathcal{J}\mathcal{U}_r(\mathcal{B})$ and $\mathcal{D}\mathcal{U}_r(\mathcal{A} \cap \mathcal{B}) = \emptyset \neq \{\rho_2\} = \mathcal{D}\mathcal{U}_r(\mathcal{A}) \cap \mathcal{D}\mathcal{U}_r(\mathcal{B})$.

Let $\mathcal{C} = \{\rho_3\}$. We obtain $\mathcal{J}\mathcal{U}_r(\mathcal{B} \cup \mathcal{C}) = \Sigma \neq \{\rho_2, \rho_3, \rho_4\} = \mathcal{J}\mathcal{U}_r(\mathcal{B}) \cup \mathcal{J}\mathcal{U}_r(\mathcal{C})$ and $\mathcal{D}\mathcal{U}_r(\mathcal{B} \cup \mathcal{C}) = \Sigma \neq \{\rho_2, \rho_3\} = \mathcal{D}\mathcal{U}_r(\mathcal{B}) \cup \mathcal{D}\mathcal{U}_r(\mathcal{C})$.

Let $\mathcal{E} = \{\rho_1, \rho_2\}$ and $\mathcal{F} = \{\rho_1, \rho_3\}$. We get $\mathcal{J}\mathcal{L}_r(\mathcal{E} \cap \mathcal{F}) = \emptyset \neq \{\rho_1\} = \mathcal{J}\mathcal{L}_r(\mathcal{E}) \cap \mathcal{J}\mathcal{L}_r(\mathcal{F})$ and $\mathcal{D}\mathcal{L}_r(\mathcal{E} \cap \mathcal{F}) = \emptyset \neq \{\rho_1\} = \mathcal{D}\mathcal{L}_r(\mathcal{E}) \cap \mathcal{D}\mathcal{L}_r(\mathcal{F})$.

Example 4.5. Consider Example 3.6. We get $\mathcal{J}\mathcal{L}_r(\Sigma) = \{\rho_2, \rho_3, \rho_4\} \neq \Sigma$ and $\mathcal{D}\mathcal{U}_r(\emptyset) = \{\rho_1\} \neq \emptyset$.

Let $\mathcal{E} = \{\rho_1, \rho_2\}$. We obtain $\mathcal{J}\mathcal{L}_r(\mathcal{J}\mathcal{U}_r(\mathcal{E})) = \{\rho_2\} \neq \{\rho_1, \rho_2\} = \mathcal{J}\mathcal{U}_r(\mathcal{E})$ and $\mathcal{D}\mathcal{L}_r(\mathcal{D}\mathcal{U}_r(\mathcal{E})) = \{\rho_2\} \neq \{\rho_1, \rho_2\} = \mathcal{D}\mathcal{U}_r(\mathcal{E})$.

Let $\mathcal{G} = \{\rho_3, \rho_4\}$. We get $\mathcal{J}\mathcal{L}_r(\mathcal{J}\mathcal{U}_r(\mathcal{G})) = \mathcal{G} \neq \{\rho_1, \rho_3, \rho_4\} = \mathcal{J}\mathcal{U}_r(\mathcal{J}\mathcal{L}_r(\mathcal{G}))$ and $\mathcal{D}\mathcal{L}_r(\mathcal{J}\mathcal{U}_r(\mathcal{G})) = \mathcal{G} \neq \{\rho_1, \rho_3, \rho_4\} = \mathcal{D}\mathcal{U}_r(\mathcal{J}\mathcal{L}_r(\mathcal{G}))$.

The following remark illustrates the relationships between ordered j -approximation spaces and ordered approximation spaces introduced by Shalil et al. [17] and the second type of approximations defined by Yao [20] and Atef et al. [9], under certain binary relations. Consequently, ordered j -approximation spaces can be viewed as a generalization of them.

Remark 4.6.

1. If $\S$ is an equivalence relation, then Definition 4.2 coincides with Definition 2.11.
2. If $\S$ is any binary relation and $\preceq$ is an equality relation, then $\mathcal{J}\mathcal{L}_j(\Upsilon) = \mathcal{D}\mathcal{L}_j(\Upsilon)$ and $\mathcal{J}\mathcal{U}_j(\Upsilon) = \mathcal{D}\mathcal{U}_j(\Upsilon)$ hold for each $j \in J$ and for all $\Upsilon \subseteq \Sigma$. In addition, Definition 4.2 coincides with Definition 2.5 for $j = r$ and Definition 2.9 for each $j \in \{l, i, u\}$.

Proposition 4.7. Let $(\Sigma, \mathcal{N}_j, \preceq)$ be a j - $\mathcal{OAS}$. Then, for each $j \in J$ and for all $\Upsilon \subseteq \Sigma$, the following statements hold:

- (1) $\mathcal{N}_{j\&}(\Upsilon) \subseteq \mathcal{I}\mathcal{L}_j(\Upsilon)$ and $\mathcal{N}_{j\&}(\Upsilon) \subseteq \mathcal{D}\mathcal{L}_j(\Upsilon)$.
- (2) $\mathcal{I}\mathcal{U}_j(\Upsilon) \subseteq \mathcal{N}_j^{\&}(\Upsilon)$ and $\mathcal{D}\mathcal{U}_j(\Upsilon) \subseteq \mathcal{N}_j^{\&}(\Upsilon)$.

Proof.

- (1) We have $\mathcal{N}_{j\&}(\Upsilon) \subseteq \Upsilon$ by Proposition 3.16. (2) in [9]. This implies $\mathcal{N}_{j\&}(\Upsilon) = \cup\{\mathcal{N}_j(x) : \mathcal{N}_j(x) \subseteq \Upsilon\} \cap \Upsilon \subseteq \cup\{\mathcal{I}\mathcal{N}_j(x) : \mathcal{N}_j(x) \subseteq \Upsilon\} \cap \Upsilon = \mathcal{I}\mathcal{L}_j(\Upsilon)$ from Proposition 3.5 (1). Similarly, we get $\mathcal{N}_{j\&}(\Upsilon) \subseteq \mathcal{D}\mathcal{L}_j(\Upsilon)$ for each $j \in J$.
- (2) From (1), we have $\mathcal{N}_{j\&}(\Upsilon^c) \subseteq \mathcal{I}\mathcal{L}_j(\Upsilon^c)$ and $\mathcal{N}_{j\&}(\Upsilon^c) \subseteq \mathcal{D}\mathcal{L}_j(\Upsilon^c)$. This implies $\mathcal{N}_j^{\&}(\Upsilon) = [\mathcal{N}_{j\&}(\Upsilon^c)]^c \supseteq [\mathcal{I}\mathcal{L}_j(\Upsilon^c)]^c = \mathcal{D}\mathcal{U}_j(\Upsilon)$ and $\mathcal{N}_j^{\&}(\Upsilon) = [\mathcal{N}_{j\&}(\Upsilon^c)]^c \supseteq [\mathcal{D}\mathcal{L}_j(\Upsilon^c)]^c = \mathcal{I}\mathcal{U}_j(\Upsilon)$. $\square$

Corollary 4.8. Let $(\Sigma, \mathcal{N}_j, \preceq)$ be a j - $\mathcal{OAS}$. Then, for each $j \in J$ and for all $\Upsilon \subseteq \Sigma$, the following statements hold:

1. $\alpha_j(\Upsilon) \leq i_{\preceq} \alpha_j(\Upsilon)$ and $\alpha_j(\Upsilon) \leq d_{\preceq} \alpha_j(\Upsilon)$.
2. $\mathcal{I}\mathcal{B}_j(\Upsilon) \subseteq \mathcal{B}_j(\Upsilon)$ and $\mathcal{D}\mathcal{B}_j(\Upsilon) \subseteq \mathcal{B}_j(\Upsilon)$.

As demonstrated by Proposition 4.7 and Corollary 4.8, with respect to the approaches of Yao [20] and Atef et al. [9], our method more effectively reduces the lower approximations and enlarges the upper approximations, thereby shrinking the boundary and increasing accuracy.

Example 4.9. Let $\preceq = \triangle \cup \{(\rho_1, \rho_2), (\rho_1, \rho_3), (\rho_1, \rho_4), (\rho_2, \rho_4), (\rho_3, \rho_2), (\rho_3, \rho_4)\}$ be a partial order relation and let $\& = \{(\rho_1, \rho_1), (\rho_1, \rho_2), (\rho_2, \rho_3), (\rho_2, \rho_4), (\rho_3, \rho_4), (\rho_4, \rho_1), (\rho_4, \rho_2)\}$ be a relation on $\Sigma = \{\rho_1, \rho_2, \rho_3, \rho_4\}$.

Table 2 is generated using the increasing and decreasing r -approximation operators together with the second type of right approximation operators. As seen in Table 2, the boundary of $\{\rho_1, \rho_2, \rho_3\}$ is $\{\rho_3\}$ under Yao's approach, whereas it is $\emptyset$ under our approaches. In this case, $\{\rho_1, \rho_2, \rho_3\}$ is a rough set according to Yao's approach, whereas it is an increasing and decreasing r -defined set under our approaches.

Table 3 is generated using the increasing and decreasing l -approximation operators together with the second type of left approximation operators. As seen in Table 3, the boundary of $\{\rho_1, \rho_3, \rho_4\}$ is $\{\rho_3\}$ under the approach of Atef et al., whereas it is $\emptyset$ under our approaches. In this case, $\{\rho_1, \rho_3, \rho_4\}$ is a rough set according to Atef et al.'s approach but an increasing and decreasing l -defined set under our approaches.

Table 2: Comparison of present methods and Yao's method [20].

	Yao's method			Our method with increasing			Our method with decreasing		
$\Upsilon \subseteq \Sigma$	$\mathcal{N}_{\&}(\Upsilon)$	$\mathcal{N}_r^{\&}(\Upsilon)$	$\mathcal{B}_r(\Upsilon)$	$\mathcal{I}\mathcal{L}_r(\Upsilon)$	$\mathcal{I}\mathcal{U}_r(\Upsilon)$	$\mathcal{I}\mathcal{B}_r(\Upsilon)$	$\mathcal{D}\mathcal{L}_r(\Upsilon)$	$\mathcal{D}\mathcal{U}_r(\Upsilon)$	$\mathcal{D}\mathcal{B}_r(\Upsilon)$
$\{\rho_1\}$	$\emptyset$	$\{\rho_1, \rho_2\}$	$\{\rho_1, \rho_2\}$	$\emptyset$	$\{\rho_1\}$	$\{\rho_1\}$	$\emptyset$	$\{\rho_1\}$	$\{\rho_1\}$
$\{\rho_2\}$	$\emptyset$	$\{\rho_1, \rho_2\}$	$\{\rho_1, \rho_2\}$	$\emptyset$	$\{\rho_2\}$	$\{\rho_2\}$	$\emptyset$	$\{\rho_1, \rho_2\}$	$\{\rho_1, \rho_2\}$
$\{\rho_3\}$	$\emptyset$	$\{\rho_3\}$	$\{\rho_3\}$	$\emptyset$	$\{\rho_3\}$	$\{\rho_3\}$	$\emptyset$	$\{\rho_3\}$	$\{\rho_3\}$
$\{\rho_4\}$	$\{\rho_4\}$	$\{\rho_3, \rho_4\}$	$\{\rho_3\}$	$\{\rho_4\}$	$\{\rho_4\}$	$\emptyset$	$\{\rho_4\}$	$\{\rho_4\}$	$\emptyset$
$\{\rho_1, \rho_2\}$	$\{\rho_1, \rho_2\}$	$\{\rho_1, \rho_2\}$	$\emptyset$	$\{\rho_1, \rho_2\}$	$\{\rho_1, \rho_2\}$	$\emptyset$	$\{\rho_1, \rho_2\}$	$\{\rho_1, \rho_2\}$	$\emptyset$
$\{\rho_1, \rho_3\}$	$\emptyset$	$\{\rho_1, \rho_2, \rho_3\}$	$\{\rho_1, \rho_2, \rho_3\}$	$\emptyset$	$\{\rho_1, \rho_3\}$	$\{\rho_1, \rho_3\}$	$\emptyset$	$\{\rho_1, \rho_2, \rho_3\}$	$\{\rho_1, \rho_2, \rho_3\}$
$\{\rho_1, \rho_4\}$	$\{\rho_4\}$	Σ	$\{\rho_1, \rho_2, \rho_3\}$	$\{\rho_4\}$	Σ	$\{\rho_1, \rho_2, \rho_3\}$	$\{\rho_1, \rho_4\}$	Σ	$\{\rho_2, \rho_3\}$
$\{\rho_2, \rho_3\}$	$\emptyset$	$\{\rho_1, \rho_2, \rho_3\}$	$\{\rho_1, \rho_2, \rho_3\}$	$\emptyset$	$\{\rho_2, \rho_3\}$	$\{\rho_2, \rho_3\}$	$\emptyset$	$\{\rho_1, \rho_2, \rho_3\}$	$\{\rho_1, \rho_2, \rho_3\}$
$\{\rho_2, \rho_4\}$	$\{\rho_4\}$	Σ	$\{\rho_1, \rho_2, \rho_3\}$	$\{\rho_4\}$	Σ	$\{\rho_1, \rho_2, \rho_3\}$	$\{\rho_2, \rho_4\}$	Σ	$\{\rho_1, \rho_3\}$
$\{\rho_3, \rho_4\}$	$\{\rho_3, \rho_4\}$	$\{\rho_3, \rho_4\}$	$\emptyset$	$\{\rho_3, \rho_4\}$	$\{\rho_3, \rho_4\}$	$\emptyset$	$\{\rho_3, \rho_4\}$	$\{\rho_3, \rho_4\}$	$\emptyset$
$\{\rho_1, \rho_2, \rho_3\}$	$\{\rho_1, \rho_2\}$	$\{\rho_1, \rho_2, \rho_3\}$	$\{\rho_3\}$	$\{\rho_1, \rho_2, \rho_3\}$	$\{\rho_1, \rho_2, \rho_3\}$	$\emptyset$	$\{\rho_1, \rho_2, \rho_3\}$	$\{\rho_1, \rho_2, \rho_3\}$	$\emptyset$
$\{\rho_1, \rho_2, \rho_4\}$	$\{\rho_1, \rho_2, \rho_4\}$	Σ	$\{\rho_3\}$	$\{\rho_1, \rho_2, \rho_4\}$	Σ	$\{\rho_3\}$	$\{\rho_1, \rho_2, \rho_4\}$	Σ	$\{\rho_3\}$
$\{\rho_1, \rho_3, \rho_4\}$	$\{\rho_3, \rho_4\}$	Σ	$\{\rho_1, \rho_2\}$	$\{\rho_3, \rho_4\}$	Σ	$\{\rho_1, \rho_2\}$	$\{\rho_1, \rho_3, \rho_4\}$	Σ	$\{\rho_2\}$
$\{\rho_2, \rho_3, \rho_4\}$	$\{\rho_3, \rho_4\}$	Σ	$\{\rho_1, \rho_2\}$	$\{\rho_2, \rho_3, \rho_4\}$	Σ	$\{\rho_1\}$	$\{\rho_2, \rho_3, \rho_4\}$	Σ	$\{\rho_1\}$

Remark 4.10. The increasing and decreasing j -lower approximations are not comparable. This can be seen from Example 4.9. Let $\Upsilon = \{\rho_1, \rho_2, \rho_3\}$ and $\Gamma = \{\rho_2, \rho_4\}$. From Table 3, we have $\mathcal{I}\mathcal{L}_l(\Upsilon) = \{\rho_2, \rho_3\} \subseteq \{\rho_1, \rho_2, \rho_3\} = \mathcal{D}\mathcal{L}_l(\Upsilon)$ and $\mathcal{D}\mathcal{L}_l(\Gamma) = \{\rho_2\} \subseteq \Gamma = \mathcal{I}\mathcal{L}_l(\Gamma)$. Similarly, the increasing and decreasing j -upper approximations are not comparable.

Proposition 4.11. Let $(\Sigma, \mathcal{N}_j, \preceq)$ be a j - $\mathcal{OAS}$. If $\&$ is an equality relation, then $\mathcal{I}\mathcal{L}_j(\Upsilon) = \Upsilon = \mathcal{I}\mathcal{U}_j(\Upsilon)$ and $\mathcal{D}\mathcal{L}_j(\Upsilon) = \Upsilon = \mathcal{D}\mathcal{U}_j(\Upsilon)$ hold for all $\Upsilon \subseteq \Sigma$.

Proof. Let $\&$ be an equality relation. We have $\mathcal{I}\mathcal{N}_j(x) = i_{\preceq}(x)$ for each $j \in J$ from Proposition 3.7 (1). It follows that $\mathcal{I}\mathcal{L}_j(\Upsilon) = \cup\{\mathcal{I}\mathcal{N}_j(x) : \mathcal{N}_j(x) \subseteq \Upsilon\} \cap \Upsilon = \cup\{i_{\preceq}(x) : x \in \Upsilon\} \cap \Upsilon = \Upsilon$ since $\preceq$ is a reflexive relation. Similarly, we get $\mathcal{D}\mathcal{L}_j(\Upsilon) = \Upsilon$ from Proposition 3.7 (1). Also, it is clear that $\mathcal{I}\mathcal{U}_j(\Upsilon) = \Upsilon = \mathcal{D}\mathcal{U}_j(\Upsilon)$ from the definitions of ordered j -upper approximations. $\square$

Corollary 4.12. Let $(\Sigma, \mathcal{N}_j, \preceq)$ be a j - $\mathcal{OAS}$. If $\&$ is an equality relation, then the equalities of the properties (IJ1), (IJ5)-(IJ8), (IJ11)-(IJ12) (and similarly (DJ1), (DJ5)-(DJ8), (DJ11)-(DJ12)) in Proposition 4.3 hold.

Proposition 4.13. Let $(\Sigma, \mathcal{N}_j, \preceq)$ be a j - $\mathcal{OAS}$. If $\&$ is a reflexive relation, then $\mathcal{I}\mathcal{L}_j(\Sigma) = \mathcal{D}\mathcal{L}_j(\Sigma) = \Sigma$ and $\mathcal{I}\mathcal{U}_j(\emptyset) = \mathcal{D}\mathcal{U}_j(\emptyset) = \emptyset$.

Table 3: Comparison of present methods and Atef et al.'s method [9].

	Atef et al.'s method			Our method with increasing			Our method with decreasing		
$\Upsilon \subseteq \Sigma$	$\mathcal{N}_j(\Upsilon)$	$\mathcal{N}_j^{\S}(\Upsilon)$	$\mathcal{B}_l(\Upsilon)$	$\mathcal{JL}_l(\Upsilon)$	$\mathcal{JU}_l(\Upsilon)$	$\mathcal{JB}_l(\Upsilon)$	$\mathcal{DL}_l(\Upsilon)$	$\mathcal{DU}_l(\Upsilon)$	$\mathcal{DB}_l(\Upsilon)$
$\{\rho_1\}$	$\emptyset$	$\{\rho_1, \rho_4\}$	$\{\rho_1, \rho_4\}$	$\emptyset$	$\{\rho_1, \rho_4\}$	$\{\rho_1, \rho_4\}$	$\emptyset$	$\{\rho_1\}$	$\{\rho_1\}$
$\{\rho_2\}$	$\{\rho_2\}$	$\{\rho_2, \rho_3\}$	$\{\rho_3\}$	$\{\rho_2\}$	$\{\rho_2\}$	$\emptyset$	$\{\rho_2\}$	$\{\rho_2\}$	$\emptyset$
$\{\rho_3\}$	$\emptyset$	$\{\rho_3\}$	$\{\rho_3\}$	$\emptyset$	$\{\rho_3\}$	$\{\rho_3\}$	$\emptyset$	$\{\rho_3\}$	$\{\rho_3\}$
$\{\rho_4\}$	$\emptyset$	$\{\rho_1, \rho_4\}$	$\{\rho_1, \rho_4\}$	$\emptyset$	$\{\rho_4\}$	$\{\rho_4\}$	$\emptyset$	$\{\rho_1, \rho_4\}$	$\{\rho_1, \rho_4\}$
$\{\rho_1, \rho_2\}$	$\{\rho_2\}$	Σ	$\{\rho_1, \rho_3, \rho_4\}$	$\{\rho_2\}$	Σ	$\{\rho_1, \rho_3, \rho_4\}$	$\{\rho_1, \rho_2\}$	Σ	$\{\rho_3, \rho_4\}$
$\{\rho_1, \rho_3\}$	$\emptyset$	$\{\rho_1, \rho_3, \rho_4\}$	$\{\rho_1, \rho_3, \rho_4\}$	$\emptyset$	$\{\rho_1, \rho_3, \rho_4\}$	$\{\rho_1, \rho_3, \rho_4\}$	$\emptyset$	$\{\rho_1, \rho_3\}$	$\{\rho_1, \rho_3\}$
$\{\rho_1, \rho_4\}$	$\{\rho_1, \rho_4\}$	$\{\rho_1, \rho_4\}$	$\emptyset$	$\{\rho_1, \rho_4\}$	$\{\rho_1, \rho_4\}$	$\emptyset$	$\{\rho_1, \rho_4\}$	$\{\rho_1, \rho_4\}$	$\emptyset$
$\{\rho_2, \rho_3\}$	$\{\rho_2, \rho_3\}$	$\{\rho_2, \rho_3\}$	$\emptyset$	$\{\rho_2, \rho_3\}$	$\{\rho_2, \rho_3\}$	$\emptyset$	$\{\rho_2, \rho_3\}$	$\{\rho_2, \rho_3\}$	$\emptyset$
$\{\rho_2, \rho_4\}$	$\{\rho_2\}$	Σ	$\{\rho_1, \rho_3, \rho_4\}$	$\{\rho_2, \rho_4\}$	Σ	$\{\rho_1, \rho_3\}$	$\{\rho_2\}$	Σ	$\{\rho_1, \rho_3, \rho_4\}$
$\{\rho_3, \rho_4\}$	$\emptyset$	$\{\rho_1, \rho_3, \rho_4\}$	$\{\rho_1, \rho_3, \rho_4\}$	$\emptyset$	$\{\rho_3, \rho_4\}$	$\{\rho_3, \rho_4\}$	$\emptyset$	$\{\rho_1, \rho_3, \rho_4\}$	$\{\rho_1, \rho_3, \rho_4\}$
$\{\rho_1, \rho_2, \rho_3\}$	$\{\rho_2, \rho_3\}$	Σ	$\{\rho_1, \rho_4\}$	$\{\rho_2, \rho_3\}$	Σ	$\{\rho_1, \rho_4\}$	$\{\rho_1, \rho_2, \rho_3\}$	Σ	$\{\rho_4\}$
$\{\rho_1, \rho_2, \rho_4\}$	$\{\rho_1, \rho_2, \rho_4\}$	Σ	$\{\rho_3\}$	$\{\rho_1, \rho_2, \rho_4\}$	Σ	$\{\rho_3\}$	$\{\rho_1, \rho_2, \rho_4\}$	Σ	$\{\rho_3\}$
$\{\rho_1, \rho_3, \rho_4\}$	$\{\rho_1, \rho_4\}$	$\{\rho_1, \rho_3, \rho_4\}$	$\{\rho_3\}$	$\{\rho_1, \rho_3, \rho_4\}$	$\{\rho_1, \rho_3, \rho_4\}$	$\emptyset$	$\{\rho_1, \rho_3, \rho_4\}$	$\{\rho_1, \rho_3, \rho_4\}$	$\emptyset$
$\{\rho_2, \rho_3, \rho_4\}$	$\{\rho_2, \rho_3\}$	Σ	$\{\rho_1, \rho_4\}$	$\{\rho_2, \rho_3, \rho_4\}$	Σ	$\{\rho_1\}$	$\{\rho_2, \rho_3\}$	Σ	$\{\rho_1, \rho_4\}$

Proof. Let $\S$ be a reflexive relation. We have $x \in \mathcal{JN}_j(x)$ for each $j \in J$ from Proposition 3.9 (1). It follows that $\mathcal{JL}_j(\Sigma) = \cup\{\mathcal{JN}_j(x) : \mathcal{N}_j(x) \subseteq \Sigma\} \cap \Sigma = \Sigma$. Similarly, we get $\mathcal{DL}_j(\Sigma) = \Sigma$. By the definitions of ordered j -upper approximations, we obtain $\mathcal{JU}_j(\emptyset) = \mathcal{DU}_j(\emptyset) = \emptyset$. $\square$

Proposition 4.14. Let $(\Sigma, \mathcal{N}_j, \preceq)$ be a j - $\mathcal{OAS}$. If $\S$ is a symmetric relation, then the following statements hold for $\Upsilon \subseteq \Sigma$:

- (1) $\mathcal{JL}_r(\Upsilon) = \mathcal{JL}_l(\Upsilon) = \mathcal{JL}_i(\Upsilon) = \mathcal{JL}_u(\Upsilon)$
- (2) $\mathcal{DL}_r(\Upsilon) = \mathcal{DL}_l(\Upsilon) = \mathcal{DL}_i(\Upsilon) = \mathcal{DL}_u(\Upsilon)$
- (3) $\mathcal{JU}_r(\Upsilon) = \mathcal{JU}_l(\Upsilon) = \mathcal{JU}_i(\Upsilon) = \mathcal{JU}_u(\Upsilon)$
- (4) $\mathcal{DU}_r(\Upsilon) = \mathcal{DU}_l(\Upsilon) = \mathcal{DU}_i(\Upsilon) = \mathcal{DU}_u(\Upsilon)$

Proof. These equalities follow directly from Proposition 3.11. $\square$

In any j - $\mathcal{OAS}$, where the relation $\S$ is assumed to be a serial or inverse serial relation, Proposition 3.12 and Proposition 3.14 show that the ordered right and left neighborhood classes are non-empty. However, this is not necessarily the case for ordered right and left lower approximations of a set, as illustrated in the following example.

Example 4.15. Consider Example 3.6. Let $\mathcal{A} = \{\rho_1\}$. Although $\S$ is a serial relation, we have $\mathcal{JL}_r(\mathcal{A}) = \mathcal{DL}_r(\mathcal{A}) = \emptyset$. If, for the same example, the inverse serial relation $\S = \{(\rho_2, \rho_3), (\rho_2, \rho_4), (\rho_4, \rho_1), (\rho_4, \rho_2)\}$ is considered, then we obtain $\mathcal{JL}_l(\mathcal{A}) = \mathcal{DL}_l(\mathcal{A}) = \emptyset$.

In the following theorems, we use ordered j -approximation spaces to construct new GN topologies.

Theorem 4.16. Let $(\Sigma, \mathcal{N}_j, \preceq)$ be a j - $\mathcal{OAS}$. Then, the family $\mathfrak{S}_{\mathcal{N}_j}^I = \{\Sigma, \emptyset, \mathcal{JL}_j(\Upsilon), \mathcal{JU}_j(\Upsilon), \mathcal{JB}_j(\Upsilon)\}$ forms a GN topology for each $j \in J$ and for every subset $\Upsilon \subseteq \Sigma$.

Proof. From Proposition 4.3 (IJ1), $\mathcal{JL}_j(\Upsilon) \subseteq \Upsilon \subseteq \mathcal{JU}_j(\Upsilon)$.

- i) It is obvious that $\Sigma, \emptyset \in \mathfrak{S}_{\mathcal{N}_j}^I$.
- ii) The union of any subcollection of elements in $\mathfrak{S}_{\mathcal{N}_j}^I$ is also in element of $\mathfrak{S}_{\mathcal{N}_j}^I$ since $\mathfrak{S}_{\mathcal{N}_j}^I$ is closed under an arbitrary union by Proposition 4.3 (IJ1) and definition of increasing j -boundary.
- iii) The intersection of any finite subcollection of elements in $\mathfrak{S}_{\mathcal{N}_j}^I$ is also in element of $\mathfrak{S}_{\mathcal{N}_j}^I$ since $\mathfrak{S}_{\mathcal{N}_j}^I$ is closed under a finite intersection by Proposition 4.3 (IJ1) and definition of increasing j -boundary. $\square$

Theorem 4.17. Let $(\Sigma, \mathcal{N}_j, \preceq)$ be a j - $\mathcal{OAS}$. Then, the family $\mathfrak{S}_{\mathcal{N}_j}^D = \{\Sigma, \emptyset, \mathcal{DL}_j(\Upsilon), \mathcal{DU}_j(\Upsilon), \mathcal{DB}_j(\Upsilon)\}$ forms a GN topology for each $j \in J$ and for every subset $\Upsilon \subseteq \Sigma$.

Proof. The proof follows directly from Proposition 4.3 (DJ1), similar to the proof of Theorem 4.16. $\square$

Since every element of the GN topology can be represented as a union of elements in the basis, the following corollaries follow.

Corollary 4.18. $\mathfrak{B}_{\mathcal{N}_j}^I = \{\Sigma, \mathcal{JL}_j(\Upsilon), \mathcal{JB}_j(\Upsilon)\}$ is the basis of $\mathfrak{S}_{\mathcal{N}_j}^I$.

Corollary 4.19. $\mathfrak{B}_{\mathcal{N}_j}^D = \{\Sigma, \mathcal{DL}_j(\Upsilon), \mathcal{DB}_j(\Upsilon)\}$ is the basis of $\mathfrak{S}_{\mathcal{N}_j}^D$.

Remark 4.20. The second type of approximations defined by Atef et al. and Yao satisfy the necessary condition for constructing a GN topology for any binary relation, just as our approaches do. The following example shows that although the ordered j -approximations are comparable to those approximations of Yao and Atef et al., the GN topologies constructed by these approaches are not.

Example 4.21. Consider Example 4.9. Let $\Upsilon = \{\rho_2, \rho_3, \rho_4\}$. We obtain $\tau_{GNr} = \{\Sigma, \emptyset, \{\rho_1, \rho_2\}, \{\rho_3, \rho_4\}\}$ by Yao's method and $\mathfrak{S}_{\mathcal{N}_j}^I = \{\Sigma, \emptyset, \{\rho_1\}, \{\rho_2, \rho_3, \rho_4\}\}$ by our method, which are independent topologies. In addition, using the method of Atef et al., we obtain $\tau_{GNl} = \{\Sigma, \emptyset, \{\rho_1, \rho_4\}, \{\rho_2, \rho_3\}\}$, whereas our method yields $\mathfrak{S}_{\mathcal{N}_j}^I = \{\Sigma, \emptyset, \{\rho_1\}, \{\rho_2, \rho_3, \rho_4\}\}$. These two topologies are not comparable.

5. Application of Ordered j-Approximation Spaces for Predicting of a Lung Cancer Disease

Table 4: An information system of female patients for lung cancer [10, 11]

U	W	S	C	P	B	Age	Decision
ρ_2	03.7	40	50	40	44	60	Y
ρ_4	04	44	55	45	54	45	N
ρ_5	04.1	38	60	48	49	48	N
ρ_{10}	04.4	55	53	57	47	55	Y
ρ_{12}	04.6	60	57	59	50	72	Y
ρ_{13}	03.7	37	47	16	44	45	N
ρ_{14}	03.6	39	50	19	50	61	Y
ρ_{15}	04.1	42	51	30	47	59	Y
ρ_{17}	05	45	49	40	54	48	N
ρ_{21}	04.1	53	60	22	53	73	Y
ρ_{23}	04.5	40	48	27	50	60	Y
ρ_{24}	04.2	60	49	29	49	66	Y
ρ_{31}	03.8	41	47	28	49	71	N
ρ_{34}	03.9	53	47	55	49	56	Y
ρ_{35}	04.7	54	49	56	53	55	Y

Medical diagnosis requires accurate tools for effective decision-making. Therefore, we applied our proposed method to the diagnosis of lung cancer. The study was conducted in the Respiratory Department of Nanjing Chest Hospital, China, and involved 190 patients described by 66 attributes, as detailed in previous studies [10, 11]. Additionally, we present an application of our GN topologies using the information system in [10, 11], which includes six attributes related to lung cancer for thirty female patients: weight loss (W), shortness of breath (S), chest pain (C), persistent cough (P), blood in sputum (B), and age. The columns of Table 4 correspond to the symptoms of lung cancer, while the rows represent individual female patients. Each entry in the table provides specific information about the symptoms exhibited by each patient.

In Table 4, let $\Sigma = \{\rho_2, \rho_4, \rho_5, \rho_{10}, \rho_{12}, \rho_{13}, \rho_{14}, \rho_{15}, \rho_{17}, \rho_{21}, \rho_{23}, \rho_{24}, \rho_{31}, \rho_{34}, \rho_{35}\}$ denote the set of female patients, and let $\mathcal{AT} = \{W, S, C, P, B, Age\}$ be the set of attributes associated with the condition. The decision set is $D = \{Y(\text{yes}), N(\text{no})\}$. Let $Y = \{\rho_2, \rho_{10}, \rho_{12}, \rho_{14}, \rho_{15}, \rho_{21}, \rho_{23}, \rho_{24}, \rho_{34}, \rho_{35}\}$ denote the set of female patients with lung cancer, and let $Y^c = \Lambda = \{\rho_4, \rho_5, \rho_{13}, \rho_{17}\}$ be the set of female patients without lung cancer. For any pair of patients $\rho_a, \rho_b \in \Sigma$, the relation is defined as " $\rho_a \mathcal{R} \rho_b \iff \mathcal{V}_{\mathcal{AT}}(\rho_a) \preceq \mathcal{V}_{\mathcal{AT}}(\rho_b)$ " and the partial order relation is defined as " $\rho_a \preceq \rho_b \iff \mathcal{V}_{\mathcal{AT}}(\rho_a) \leq \mathcal{V}_{\mathcal{AT}}(\rho_b)$ " where $\mathcal{V}_{\mathcal{AT}}(\rho)$ denotes the attribute values for patient ρ .

In the first case, for patients with cancer, and in the second case, for patients without cancer, calculations are performed according to the algorithm provided at the end of this section. Based on the relations and partial order relations obtained from each attribute, bases are constructed, and the core is determined according to the changes in these bases.

Case I: The set of female patients with lung cancer

Step 1: Initial Attribute Set

Let $\mathcal{R}_{\mathcal{AT}}$ be the relation and $\preceq_{\mathcal{AT}}$ be the partial order relation on Σ with respect to $\mathcal{AT}$. The increasing r -lower and r -upper approximations of Y with respect to $\mathcal{R}_{\mathcal{AT}}$ and $\preceq_{\mathcal{AT}}$ are given by $\mathcal{IL}_{r, \mathcal{AT}}(Y) = \{\rho_{12}, \rho_{21}\}$ and $\mathcal{IU}_{r, \mathcal{AT}}(Y) = \Sigma$, respectively. Thus, the increasing r -boundary of Y is $\mathcal{IB}_{r, \mathcal{AT}}(Y) = \{\rho_{12}, \rho_{21}\}^c$. The GN topology is $\mathcal{S}_{\mathcal{R}_{\mathcal{AT}}}^I = \{\emptyset, \Sigma, \{\rho_{12}, \rho_{21}\}, \{\rho_{12}, \rho_{21}\}^c\}$ with the basis $\mathcal{B}_{\mathcal{R}_{\mathcal{AT}}}^I = \{\Sigma, \{\rho_{12}, \rho_{21}\}, \{\rho_{12}, \rho_{21}\}^c\}$. The decreasing r -lower and r -upper approximations of Y with respect to $\mathcal{R}_{\mathcal{AT}}$ and $\preceq_{\mathcal{AT}}$ are given by $\mathcal{DL}_{r, \mathcal{AT}}(Y) = \{\rho_2, \rho_{10}, \rho_{12}, \rho_{14}, \rho_{15}, \rho_{21}, \rho_{23}, \rho_{24}, \rho_{34}\} = Y \setminus \{\rho_{35}\}$ and $\mathcal{DU}_{r, \mathcal{AT}}(Y) = \Sigma$, respectively. Thus, the decreasing r -boundary of Y is $\mathcal{DB}_{r, \mathcal{AT}}(Y) = \{\rho_4, \rho_5, \rho_{13}, \rho_{17}, \rho_{35}\} = \Lambda \cup \{\rho_{35}\}$. The GN topology is $\mathcal{S}_{\mathcal{R}_{\mathcal{AT}}}^D = \{\emptyset, \Sigma, Y \setminus \{\rho_{35}\}, \Lambda \cup \{\rho_{35}\}\}$ and the corresponding basis is $\mathcal{B}_{\mathcal{R}_{\mathcal{AT}}}^D = \{\Sigma, Y \setminus \{\rho_{35}\}, \Lambda \cup \{\rho_{35}\}\}$.

Step 2: Attribute Removal: Weight Loss (W)

If the attribute "Weight loss (W)" is removed from the attribute set $\mathcal{AT}$, the approximations and boundaries remain identical to Step 1. Hence, weight loss is a reducible attribute.

Step 3: Attribute Removal: Shortness of Breath (S)

Similarly, removing the attribute "Shortness of breath (S)" from the attribute set $\mathcal{AT}$ yields identical approximations and boundaries as in Step 1. Therefore, shortness of breath is also a reducible attribute.

Step 4: Attribute Removal: Chest Pain (C)

Now consider the case where the attribute "Chest pain (C)" is removed. Let $\mathcal{R}_{\mathcal{AT} \setminus \{C\}}$ denote the relation, and let $\preceq_{\mathcal{AT} \setminus \{C\}}$ be the partial order relation on Σ with respect to $\mathcal{AT} \setminus \{C\}$. The increasing r -lower and r -upper approximations of Y with respect to $\mathcal{R}_{\mathcal{AT} \setminus \{C\}}$ and $\preceq_{\mathcal{AT} \setminus \{C\}}$ are given by $\mathcal{IL}_{r, \mathcal{AT} \setminus \{C\}}(Y) = \{\rho_{12}, \rho_{21}, \rho_{35}\}$ and $\mathcal{IU}_{r, \mathcal{AT} \setminus \{C\}}(Y) = \Sigma$, respectively. Thus, the increasing r -boundary of Y is $\mathcal{IB}_{r, \mathcal{AT} \setminus \{C\}}(Y) = \{\rho_{12}, \rho_{21}, \rho_{35}\}^c$. The GN topology is $\mathcal{S}_{\mathcal{R}_{\mathcal{AT} \setminus \{C\}}}^I = \{\emptyset, \Sigma, \{\rho_{12}, \rho_{21}, \rho_{35}\}, \{\rho_{12}, \rho_{21}, \rho_{35}\}^c\}$ and its basis is $\mathcal{B}_{\mathcal{R}_{\mathcal{AT} \setminus \{C\}}}^I = \{\Sigma, \{\rho_{12}, \rho_{21}, \rho_{35}\}, \{\rho_{12}, \rho_{21}, \rho_{35}\}^c\}$. The decreasing r -lower and r -upper approximations of Y with respect to $\mathcal{R}_{\mathcal{AT} \setminus \{C\}}$ and $\preceq_{\mathcal{AT} \setminus \{C\}}$ are given by $\mathcal{DL}_{r, \mathcal{AT} \setminus \{C\}}(Y) = Y$ and $\mathcal{DU}_{r, \mathcal{AT} \setminus \{C\}}(Y) = \Sigma$, respectively. Hence, the decreasing r -boundary of Y is $\mathcal{DB}_{r, \mathcal{AT} \setminus \{C\}}(Y) = \Lambda$. The GN topology is $\mathcal{S}_{\mathcal{R}_{\mathcal{AT} \setminus \{C\}}}^D = \{\emptyset, \Sigma, Y, \Lambda\}$ and its basis is $\mathcal{B}_{\mathcal{R}_{\mathcal{AT} \setminus \{C\}}}^D = \{\Sigma, Y, \Lambda\}$.

Since the basis changes when attribute C is removed, i.e., $\mathfrak{B}_{\mathcal{N}, \mathcal{A}\mathcal{T} \setminus \{C\}}^I \neq \mathfrak{B}_{\mathcal{N}, \mathcal{A}\mathcal{T}}^I$ and $\mathfrak{B}_{\mathcal{N}, \mathcal{A}\mathcal{T} \setminus \{C\}}^D \neq \mathfrak{B}_{\mathcal{N}, \mathcal{A}\mathcal{T}}^D$, we conclude that chest pain is an irreducible attribute.

Step 5: Attribute Removal: Persistence Cough (P)

Similar to the previous steps, removing the attribute “Persistent Cough (P)” from the attribute set $\mathcal{A}\mathcal{T}$ yields identical approximations and boundaries as in Step 1. Therefore, persistence cough is a reducible attribute.

Step 6: Attribute Removal: Blood in Sputum (B)

As in Step 1, removing the attribute “Blood in sputum (B)” from the set $\mathcal{A}\mathcal{T}$ leads to identical results, confirming that blood in sputum is a reducible attribute.

Step 7: Attribute Removal: Age

Lastly, consider the removal of the attribute “Age” from $\mathcal{A}\mathcal{T}$, $\mathfrak{S}_{\mathcal{A}\mathcal{T} \setminus \{Age\}}$ be the relation and let $\preceq_{\mathcal{A}\mathcal{T} \setminus \{Age\}}$ be the partial order relation on Σ with respect to the attribute set $\mathcal{A}\mathcal{T} \setminus \{Age\}$. The increasing r -lower and r -upper approximations of Υ with respect to $\mathfrak{S}_{\mathcal{A}\mathcal{T} \setminus \{Age\}}$ and $\preceq_{\mathcal{A}\mathcal{T} \setminus \{Age\}}$ are given by $\mathcal{I}\mathcal{L}_{r, \mathcal{A}\mathcal{T} \setminus \{Age\}}(\Upsilon) = \{\rho_{12}, \rho_{35}\}$ and $\mathcal{I}\mathcal{U}_{r, \mathcal{A}\mathcal{T} \setminus \{Age\}}(\Upsilon) = \Sigma$, respectively. Therefore, the increasing r -boundary of Υ is $\mathcal{I}\mathcal{B}_{r, \mathcal{A}\mathcal{T} \setminus \{Age\}}(\Upsilon) = \{\rho_{12}, \rho_{35}\}^c$. The GN topology is $\mathfrak{S}_{\mathcal{N}, \mathcal{A}\mathcal{T} \setminus \{Age\}}^I = \{\emptyset, \Sigma, \{\rho_{12}, \rho_{35}\}, \{\rho_{12}, \rho_{35}\}^c\}$ and its basis is $\mathfrak{B}_{\mathcal{N}, \mathcal{A}\mathcal{T} \setminus \{Age\}}^I = \{\Sigma, \{\rho_{12}, \rho_{35}\}, \{\rho_{12}, \rho_{35}\}^c\}$. The decreasing r -lower and r -upper approximations of Υ with respect to $\mathfrak{S}_{\mathcal{A}\mathcal{T} \setminus \{Age\}}$ and $\preceq_{\mathcal{A}\mathcal{T} \setminus \{Age\}}$ are given by $\mathcal{D}\mathcal{L}_{r, \mathcal{A}\mathcal{T} \setminus \{Age\}}(\Upsilon) = \Upsilon \setminus \{\rho_{21}\}$ and $\mathcal{D}\mathcal{U}_{r, \mathcal{A}\mathcal{T} \setminus \{Age\}}(\Upsilon) = \Sigma$, respectively. Hence, the decreasing r -boundary of Υ is $\mathcal{D}\mathcal{B}_{r, \mathcal{A}\mathcal{T} \setminus \{Age\}}(\Upsilon) = \Lambda \cup \{\rho_{21}\}$. The GN topology is $\mathfrak{S}_{\mathcal{N}, \mathcal{A}\mathcal{T} \setminus \{Age\}}^D = \{\emptyset, \Sigma, \Upsilon \setminus \{\rho_{21}\}, \Lambda \cup \{\rho_{21}\}\}$ and its basis is $\mathfrak{B}_{\mathcal{N}, \mathcal{A}\mathcal{T} \setminus \{Age\}}^D = \{\Sigma, \Upsilon \setminus \{\rho_{21}\}, \Lambda \cup \{\rho_{21}\}\}$.

Since the basis differs after the removal of the attribute Age, i.e., $\mathfrak{B}_{\mathcal{N}, \mathcal{A}\mathcal{T} \setminus \{Age\}}^I \neq \mathfrak{B}_{\mathcal{N}, \mathcal{A}\mathcal{T}}^I$ and $\mathfrak{B}_{\mathcal{N}, \mathcal{A}\mathcal{T} \setminus \{Age\}}^D \neq \mathfrak{B}_{\mathcal{N}, \mathcal{A}\mathcal{T}}^D$, we conclude that age is an irreducible attribute.

Based on the above steps, the reducible attributes are Weight loss (W), Shortness of breath (S), Persistent cough (P), and Blood in sputum (B), while the irreducible attributes are Chest pain (C) and Age. Consequently, the core of the attribute set is $CORE(\mathcal{A}\mathcal{T}) = \{C, Age\}$ in Case I.

Case II: The set of female patients with no lung cancer

We apply the same steps as in Case I to the subset $\Lambda \subset \Sigma$, representing patients without lung cancer, and obtain the same core.

Table 5: The results for Case I: The set of female patients with lung cancer

	$\mathcal{I}\mathcal{L}_r(\Upsilon)$	$\mathcal{I}\mathcal{U}_r(\Upsilon)$	$\mathcal{D}\mathcal{L}_r(\Upsilon)$	$\mathcal{D}\mathcal{U}_r(\Upsilon)$
$\mathcal{A}\mathcal{T}$	$\{\rho_{12}, \rho_{21}\}$	Σ	$\Upsilon \setminus \{\rho_{35}\}$	Σ
$\mathcal{A}\mathcal{T} \setminus \{W\}$	$\{\rho_{12}, \rho_{21}\}$	Σ	$\Upsilon \setminus \{\rho_{35}\}$	Σ
$\mathcal{A}\mathcal{T} \setminus \{S\}$	$\{\rho_{12}, \rho_{21}\}$	Σ	$\Upsilon \setminus \{\rho_{35}\}$	Σ
$\mathcal{A}\mathcal{T} \setminus \{C\}$	$\{\rho_{12}, \rho_{21}, \rho_{35}\}$	Σ	Υ	Σ
$\mathcal{A}\mathcal{T} \setminus \{P\}$	$\{\rho_{12}, \rho_{21}\}$	Σ	$\Upsilon \setminus \{\rho_{35}\}$	Σ
$\mathcal{A}\mathcal{T} \setminus \{B\}$	$\{\rho_{12}, \rho_{21}\}$	Σ	$\Upsilon \setminus \{\rho_{35}\}$	Σ
$\mathcal{A}\mathcal{T} \setminus \{Age\}$	$\{\rho_{12}, \rho_{35}\}$	Σ	$\Upsilon \setminus \{\rho_{21}\}$	Σ

Table 6: The results for Case II: The set of female patients with no lung cancer

	$\mathcal{I}\mathcal{L}_r(\Lambda)$	$\mathcal{I}\mathcal{U}_r(\Lambda)$	$\mathcal{D}\mathcal{L}_r(\Lambda)$	$\mathcal{D}\mathcal{U}_r(\Lambda)$
$\mathcal{A}\mathcal{T}$	$\emptyset$	$\Lambda \cup \{\rho_{35}\}$	$\emptyset$	$\{\rho_{12}, \rho_{21}\}^c$
$\mathcal{A}\mathcal{T} \setminus \{W\}$	$\emptyset$	$\Lambda \cup \{\rho_{35}\}$	$\emptyset$	$\{\rho_{12}, \rho_{21}\}^c$
$\mathcal{A}\mathcal{T} \setminus \{S\}$	$\emptyset$	$\Lambda \cup \{\rho_{35}\}$	$\emptyset$	$\{\rho_{12}, \rho_{21}\}^c$
$\mathcal{A}\mathcal{T} \setminus \{C\}$	$\emptyset$	Λ	$\emptyset$	$\{\rho_{12}, \rho_{21}, \rho_{35}\}^c$
$\mathcal{A}\mathcal{T} \setminus \{P\}$	$\emptyset$	$\Lambda \cup \{\rho_{35}\}$	$\emptyset$	$\{\rho_{12}, \rho_{21}\}^c$
$\mathcal{A}\mathcal{T} \setminus \{B\}$	$\emptyset$	$\Lambda \cup \{\rho_{35}\}$	$\emptyset$	$\{\rho_{12}, \rho_{21}\}^c$
$\mathcal{A}\mathcal{T} \setminus \{Age\}$	$\emptyset$	$\Lambda \cup \{\rho_{21}\}$	$\emptyset$	$\{\rho_{12}, \rho_{35}\}^c$

As shown in Table 5 and Table 6, this finding holds for both Case I and Case II. Thus, the attributes Chest pain (C) and Age are key factors in determining whether a patient is at risk of lung cancer and should be prioritized in diagnostic tests. This highlights the significance of these attributes in clinical decision-making for the early detection of lung cancer. The application of ordered j -approximation spaces in attribute reduction, based on GN topologies, has proven effective in a real-life scenario.

Algorithm

The following algorithm enables the determination of the CORE based on the ordered j -approximations.

Step 1 Input the finite universe set Σ , a finite set of condition attributes denoted by $\mathcal{AT} = \{a_i\}_{i \in I}$, and a finite set of decision attributes denoted by $Decision$ from the information table. Take a subset $\Upsilon \subset \Sigma$.

Step 2 Define $\mathcal{S}_{\mathcal{AT}}$ and $\preceq_{\mathcal{AT}}$ with respect to $\mathcal{AT}$.

Step 3 Compute the j -neighborhoods and ordered j -neighborhood classes of each element in Σ with respect to $\mathcal{AT}$.

Step 4 Compute ordered j -lower and upper approximations and thus ordered j -boundaries for $\Upsilon \subset \Sigma$ with respect to $\mathcal{AT}$.

Step 5 Generate the bases of the GN topologies, denoted by $\mathfrak{B}_{\mathcal{N}_j \mathcal{AT}}^I$ and $\mathfrak{B}_{\mathcal{N}_j \mathcal{AT}}^D$, respectively.

Step 6 Remove an attribute a_i from $\mathcal{AT}$. Define $\mathcal{S}_{\mathcal{AT} \setminus \{a_i\}}$ and $\preceq_{\mathcal{AT} \setminus \{a_i\}}$ with respect to $\mathcal{AT} \setminus \{a_i\}$.

Step 7 Compute the j -neighborhoods and ordered j -neighborhood classes of each element in Σ with respect to $\mathcal{AT} \setminus \{a_i\}$.

Step 8 Compute ordered j -lower and upper approximations and thus ordered j -boundaries for $\Upsilon \subset \Sigma$ with respect to $\mathcal{AT} \setminus \{a_i\}$.

Step 9 Generate the bases of the GN topologies, denoted by $\mathfrak{B}_{\mathcal{N}_j \mathcal{AT} \setminus \{a_i\}}^I$ and $\mathfrak{B}_{\mathcal{N}_j \mathcal{AT} \setminus \{a_i\}}^D$, respectively.

Step 10 Repeats Step 6-9 for each $a_i \in \mathcal{AT}$.

Step 11 The attributes in $\mathcal{AT}$ for which $\mathfrak{B}_{\mathcal{N}_j \mathcal{AT}}^I \neq \mathfrak{B}_{\mathcal{N}_j \mathcal{AT} \setminus \{a_i\}}^I$ or $\mathfrak{B}_{\mathcal{N}_j \mathcal{AT}}^D \neq \mathfrak{B}_{\mathcal{N}_j \mathcal{AT} \setminus \{a_i\}}^D$ determine $CORE(\mathcal{AT})$.

6. Conclusion

Semantically, the partial order relation provides a more reliable and supportive structure for the application of real-life problems by clarifying the relationships between attributes in information systems and revealing the determinism of attributes. Moreover, in computer science, concepts such as concurrency, data dependencies and hierarchical information extraction are based on partial ordering, which significantly extends the application areas of this relation.

By using relations or j -neighborhoods instead of the equivalence relation that forms the foundation of Pawlak approximation spaces, and in contrast to the generalized approximation spaces presented in the literature, we propose a new generalized approximation space, called ordered j -approximation spaces, which is constructed by employing two flexible structures: the partial order relation and j -neighborhoods. The properties of these spaces and their relationships with earlier approximations have been thoroughly examined by us. Our approach is more effective for collapsing the upper approximation and expanding the lower approximation than the second type of approximation operators put forth by Yao [20] and Atef et al. [9]. It follows that our method provides better accuracy maximization and boundary minimization compared to these approaches. Theoretical examples and demonstrations presented in the paper support this improvement, which produces more reliable results in direct application areas.

Building on this foundation, future research will concentrate on the following areas in an effort to improve rough set theory's theoretical framework, strengthen its connections to topology, and make it more applicable to real-world issues:

- i. Generalizations: The extension of ordered j -approximation spaces using previously defined j -neighborhoods in the literature and their comparison.
- ii. Topological Structures: Analysis of the topological structures of generalized approximation spaces based on various j -neighborhoods, and ordered j -approximation spaces.
- iii. Ideal Structures: Examining the ideal structure of ordered j -approximation spaces and generalized approximation spaces building with respect to different j -neighborhoods.

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