FCMS

Fundamentals of Contemporary Mathematical Sciences
doi:10.54974/fcmathsci.1872022
(**2026**) **7**(2) 169 – 184
Research Article

Approximate Solution of Time-Fractional Kawahara Equation Using the RBF Collocation Technique

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Received: 26 January 2026 **Accepted:** 06 July 2026

Abstract: The paper presents an approximate solution to the time-fractional Kawahara equation (TFKE), marking a significant advancement in the field. We decisively utilize the Liouville-Caputo fractional derivative (FD) as the foundation of our analytical framework. To discretize the equations in time, we expertly implement the Crank-Nicolson difference scheme, which guarantees precision and effectiveness. Additionally, we conduct a comprehensive stability analysis of the present scheme, applying the von-Neumann approach designed explicitly for fractional equations. The findings from the numerical experiments are clearly illustrated in the figures, effectively showcasing the robustness and reliability of our approach.

Keywords: Fractional Kawahara equation, Crank-Nicolson method, RBF collocation technique, Liouville-Caputo fractional derivative operator, von-Neumann stability.

1. Introduction

Fractional calculus is a compelling and advanced extension of traditional calculus. In this domain, we generalize familiar integer-order derivatives and integral operators into their fractional counterparts, enabling us to compute derivatives and integrals of arbitrary order. This powerful concept not only expands our understanding but also delivers profound insights across numerous fields of science and engineering. This intriguing field has been evolving since the 17th century, capturing the imagination of researchers and gaining remarkable traction in academic discourse [27, 30, 32]. Its applications are vast and varied, illuminating its potential across a spectrum of disciplines, from physics and biology to chemistry, viscoelasticity, fluid mechanics, acoustic waves, fractal dynamics, and control theory. Each application reveals a facet of its powerful versatility, making it a topic of fervent interest in scholarly literature. A wealth of studies has delved into

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2020 *AMS Mathematics Subject Classification*: 65M12, 65N35, 65N50, 26A33

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the existence and validity of problems modeled within this compelling framework. In recent years, the integration of fractional-order derivatives into mathematical models has emerged as a powerful approach, thanks to their capability to accurately represent memory and hereditary effects found in complex systems. Consequently, fractional partial differential equations (PDEs) such as the Kawahara equation, the Fokker-Planck equation, the Rosenau equation, the KdV equation, and diffusion equations have made significant inroads into the literature, enriching the landscape of mathematical research [4, 7, 12, 20, 28]. While finding exact solutions to these equations can present formidable challenges, the quest has spurred the development of several innovative numerical methods for their resolution. Among the most widely employed approaches are finite element methods, the Crank-Nicolson technique, spectral methods, meshless Galerkin methods, moving least squares techniques, and radial basis function (RBF), and RBF collocation methods. There are many papers about the solution of fractional partial differential equations, such as [15, 16, 18, 31].

Notably, the time-fractional Kawahara equation provides an advanced framework for effectively modeling anomalous diffusion and nonlocal temporal dynamics evident in viscoelastic materials, fluid flows, and biological systems. However, the combination of significant nonlinearity and high-order spatial derivatives, along with the unique nonlocal characteristics of fractional derivatives, presents substantial challenges, often rendering analytical solutions either extremely difficult or impossible to achieve [23–25]. Embracing these complexities is essential for advancing our understanding and application of these sophisticated models [14].

As a result, developing efficient and accurate numerical methods for solving these equations has become a significant research topic. Traditional grid-based methods, such as finite difference and finite element approaches, often have limitations including mesh dependency, reduced accuracy for higher-order derivatives, and increased computational cost, particularly in higher dimensions [36]. To address these issues, we will employ a powerful alternative: a mesh-free method based on RBFs [5, 8, 17]. The RBF collocation method features high accuracy, geometric flexibility, and easy implementation without needing mesh generation [9]. Motivated by these benefits, this study aims to solve the time-fractional Kawahara equation using the RBF collocation method combined with an appropriate time discretization scheme to approximate its solutions.

The organization of this paper is structured as follows: In Section 1, we briefly present the governing equation and some recent references related to it. Section 2 discusses the application of the RBF method for obtaining numerical solutions to the aforementioned equations. In Section 4, we examine the stability of the proposed method by employing techniques similar to classical von-Neumann analysis. Section 5 includes a numerical example to demonstrate the validity of the suggested scheme, alongside several numerical experiments presented in tabular format. Finally,

we conclude with our findings and provide a list of references.

2. Statement of the Governing Equation

This part gives us the information of the mentioned time-fractional Kawahara equation (TFKE). The Kawahara equation (KE) was depicted and introduced by Kawahara. The KE is one of the effective equations in physics and ocean technology [19]. Signal dispersion across lines of transmission is indicated by this equation. Furthermore, it enhances the process by incorporating memory effects, leading to more precise formulations of waves propagating in complex temporal environments [29]. There are many numerical studies about the mentioned equation. For example, a spectral collocation algorithm is used to obtain approximate solutions of the time-fractional generalized Kawahara equation by [2]. Another study, which mentions that the numerical solution of the TFKE by using the collocation method based on a certain Lucas polynomial sequence is given by [1]. Ahmed et al. gives a novel numerical technique to solve the TFKE numerically in [3].

General form of the KE is given as follows:

$$z_t(x,t) + \alpha z(x,t) z_x(x,t) + \beta z_{xxx}(x,t) - \gamma z_{xxxxx}(x,t) = 0, \tag{1}$$

with the initilal condition

$$z(x,0) = h(x), \tag{2}$$

in here $z(x,t)$ descripts the dependent variable, t is the time variable, and x is the spatial variable. α, β and γ are real constants. The constants β and γ are dispersion parameters related to third and fifth-order spatial derivatives. The zz_x term depicts nonlinear convection, while z_{xxx} and z_{xxxxx} account for dispersive effects. The interaction between nonlinearity and dispersion plays a crucial role in shaping the behavior of solitary and periodic waves. The dispersive term represents wave dispersion; the higher-order dispersion term captures additional dispersion effects that arise in specific media, while the convective term describes the advection of the wave packet due to its own velocity [2].

Let's now consider the TFKE

$$D_t^{\delta} z(x,t) + \alpha z(x,t) z_x(x,t) + \beta z_{xxx}(x,t) - \gamma z_{xxxxx}(x,t) = f(x,t), \quad x \in [a,b], \quad t > 0, \quad 0 \le \delta < 1, \tag{3}$$

with the initial and homogeneous boundary conditions respectively,

$$z(x,0) = h(x), \quad x \in [a,b],$$

$$z(a,t) = z(b,t) = 0, \quad t > 0,$$

$$z_x(a,t) = z_x(b,t) = 0, \quad t > 0,$$

$$z_{xx}(a,t) = z_{xx}(b,t) = 0, \quad t > 0,$$

where $f(x,t)$ is a source term, $h(x)$ is a sufficiently smooth function, the operator D_t^δ represents the Liouville-Caputo FD and $z(x,t)$ is the unknown function. The fifth-order derivative z_{xxxxx} represents higher-order dispersion effects and the nonlinear term zz_x denotes wave steepening. The fractional derivative can be said to introduce memory effects into the system. The selection of the Liouville-Caputo fractional derivative in this study is driven by its physical relevance and computational convenience, especially regarding time-dependent nonlinear partial differential equations, such as the time-fractional Kawahara equation. The Liouville-Caputo FD presents compelling advantages for various applications.

First, it facilitates the formulation of initial conditions through integer-order derivatives, which offer clear and intuitive physical interpretations. This clarity is crucial for ensuring accurate modeling in practical scenarios. Second, it excels at capturing memory and hereditary effects within complex systems, making it an indispensable tool for researchers and practitioners alike. Finally, from a numerical perspective, the Liouville-Caputo FD supports stable and widely adopted discretization schemes, such as the L1 approximation and its variants, ensuring reliable results in computational analyses. These schemes are relatively straightforward to implement and integrate efficiently with spatial discretization methods like the RBF collocation method [11, 21, 30]. Ultimately, the Liouville-Caputo FD is one of the most commonly used definitions in the literature on fractional calculus. This definition facilitates comparisons with existing studies and ensures consistency with previously developed numerical methods [23, 32].

The definition of Liouville-Caputo FD is provided below:

Definition 2.1. *The Liouville-Caputo FD $D_t^\delta z(x,t)$ of order $\delta > 0$ is denoted as [32]*

$$D_t^\delta z(x,t) = \begin{cases} \frac{1}{\Gamma(m-\delta)} \int_0^t \frac{\partial^m z(x,\xi)}{\partial \xi^m} (t-\xi)^{m-\delta-1} d\xi, & m-1 < \delta < m, \\ \frac{\partial^m z(x,t)}{\partial t^m}, & \delta = m, \quad m \in \mathbb{N}. \end{cases} \tag{4}$$

This FD has the following property for $m-1 \leq \delta < m$, $m \in \mathbb{N}$,

$$D_t^\delta z(x,t) = \begin{cases} 0, & m \in \mathbb{N}_0 \quad and \quad m < \lceil \delta \rceil, \\ \frac{\Gamma(m+1)}{\Gamma(m-\delta+1)} t^{m-\delta}, & m \in \mathbb{N}_0 \quad and \quad m \geq \lceil \delta \rceil, \end{cases} \tag{5}$$

where $\mathbb{N}_0 = \{0,1,2,3,\cdots\}$, the function $\Gamma(.)$ is called a Gamma function and the notation $\lceil \delta \rceil$ denotes the ceiling function.

A useful approximation was presented by Sun and Wu [33] for computing the fractional derivative concerning t. Now, let's state the following lemma about this approximation.

Lemma 2.2. *[33] Let $z(x,t) \in C^2[0,t_l]$. Then*

$$\begin{aligned} |R(z(x,t_l))| &= \left| \frac{1}{\gamma(1-\delta)} \int_0^{t_l} \frac{z'(x,s)}{(t_l-s)^{\delta}} ds \right. \\ &- \left. \frac{\Delta t^{-\delta}}{\gamma(2-\delta)} [b_0 z(x,t_l) - \sum_{i=1}^{l-1} (b_{l-i-1} - b_{l-i}) z(x,t_i) - b_{l-1} z(x,t_0)] \right| \\ &\leq \frac{1}{\gamma(2-\delta)} \left[\frac{1-\delta}{12} + \frac{2^{2-\delta}}{2-\delta} - (1+2^{-\delta}) \right] \max_{0 \leq t \leq t_l} |z''(x,t)| \Delta t^{2-\delta}, \end{aligned}$$

where $\delta \in (0,1)$ ve $b_i = (i+1)^{1-\delta} - i^{1-\delta}$.

Let's use the L1 approximation for this derivative.

$$D_t^{\delta} z(x_j, t_{n+1}) = \mu \left[b_0 z_j^{n+1} - \sum_{i=1}^{n} (b_{n-i} - b_{n-i+1}) z_j^i - b_n z_j^0 \right] + \mathcal{O}((\Delta t)^{2-\delta}), \tag{6}$$

with

$$b_i = (i+1)^{1-\delta} - i^{1-\delta}, \tag{7}$$

$$\mu = \frac{(\Delta t)^{-\delta}}{\Gamma(2-\delta)}. \tag{8}$$

3. Implementation of the RBF Collocation Technique to the TFKE

This part introduces the RBF collocation technique. First, let's delve into the captivating concept of RBFs and uncover their significance.

Definition 3.1. *Let s be a positive integer. A RBF $\Phi : \mathbb{R}^s \to \mathbb{R}$ is introduced as*

$$\Phi(\mathbf{x}) = \phi(\|x - x_j\|), \tag{9}$$

where $\phi : [0,\infty) \to \mathbb{R}$ is a function with one variable and $\|\cdot\|$ is a norm in $\mathbb{R}^s$.

It depends on the distance from a specific center point.

This collocation technique relies on point collocation of a straightforward series expression, which simplifies the mathematical algorithm and makes it easy to implement. In comparison to classical finite difference methods, the RBF collocation approach offers several advantages: it is mesh-free, it efficiently handles high-order derivatives, it provides spectral-like spatial accuracy for smooth solutions, and it is flexible for nonlinear fractional models. Kansa's RBF collocation technique was developed to solve PDEs using RBF [17]. Among the available approximation schemes, the multiquadric approach has been favored due to its remarkable accuracy. Hardy was initially introduced to this scheme [13]. Researchers established the existence, uniqueness, and

convergence of this approximation [10, 22, 26]. Recently, there has been increasing interest in applying this function to approximate solutions for fractional PDEs. The mentioned method has been applied to solve fractional PDEs by authors [6, 35]. We shall now proceed to discuss the implementation of the RBF collocation technique in detail.

Utilizing the method, our objective is to approximate the value of $z(x_j, t^n)$:

$$z(x_j, t^n) = \sum_{k=1}^{M} \lambda_k^n \phi_k(x_j), \tag{10}$$

where λ_k^n are the unkown coefficients and $\phi_k(x_j)$ is a RBF. The most widely utilized RBFs are Gaussian (G), Multiquadric (MQ) which are presented below:

- $\phi(r) = e^{-cr^2}$,
- $\phi(r) = \sqrt{r^2 + c^2}$.

This paper presents an approach that utilizes both Gaussian basis function to derive numerical solutions.

First, we apply Liouville-Caputo FD which is given Equation (6), and then Crank-Nicolson technique to the Equation (3), we get

$$\begin{aligned} \mu\Big[b_0 z_j^{n+1} &- \sum_{i=1}^{n}(b_{n-i} - b_{n-i+1}) z_j^i - b_n z_j^0\Big] \\ &+ \alpha\frac{(z)_j^{n+1}(z_x)_j^n + (z)_j^n(z_x)_j^{n+1}}{2} + \beta\frac{(z_{3x})_j^{n+1} + (z_{3x})_j^n}{2} \\ &- \gamma\frac{(z_{5x})_j^{n+1} + (z_{5x})_j^n}{2} = 0, \end{aligned} \tag{11}$$

where $t^{n+1} = (n+1)\Delta t, \quad n = 0, 1, 2, \cdots, M$ and the step size of the time variable t is depicted by Δt. The approximate solution of $z(x, t)$ at the point (x_j, t^n) are denoted by z_j^n. The approximation of $z(x_j, t^n)$ and its spatial derivative at each time step can be stated as:

$$z(x_j, t^n) = \sum_{k=1}^{M} \lambda_k^n \phi_k(x_j), \tag{12}$$

$$\frac{\partial^3 z(x_j, t^n)}{\partial x^3} = \sum_{k=1}^{M} \lambda_k^n \phi_k'''(x_j), \tag{13}$$

$$\frac{\partial^5 z(x_j, t^n)}{\partial x^5} = \sum_{k=1}^{M} \lambda_k^n \phi_k^{(v)}(x_j), \tag{14}$$

where the coefficients, denoted as λ_k^n, need to be computed at every time step. By substituting Equations (12)-(14) into the Equation (11) and rearranging, we obtain the following algebraic equation:

$$\begin{aligned}
& \mu \sum_{k=1}^{M} \lambda_k^{n+1} \phi_k(x_j) \\
+ \quad & \frac{\alpha}{2}\left[\sum_{k=1}^{M} \lambda_k^{n+1} \phi_k(x_j) \sum_{k=1}^{M} \lambda_k^{n} \phi_k'(x_j) + \sum_{k=1}^{M} \lambda_k^{n+1} \phi_k'(x_j) \sum_{k=1}^{M} \lambda_k^{n} \phi_k(x_j) \right] \\
+ \quad & \frac{\beta}{2} \sum_{k=1}^{M} \lambda_k^{n+1} \phi_k'''(x_j) - \frac{\gamma}{2} \sum_{k=1}^{M} \lambda_k^{n+1} \phi_k^{(v)}(x_j) \\
= \quad & \mu \sum_{i=1}^{n} (b_{n-i} - b_{n-i+1}) \sum_{k=1}^{M} \lambda_k^{i} \phi_k(x_j) + \mu b_n \sum_{k=1}^{M} \lambda_k^{0} \phi_k(x_j) \\
- \quad & \frac{\beta}{2} \sum_{k=1}^{M} \lambda_k^{n} \phi_k'''(x_j) + \frac{\gamma}{2} \sum_{k=1}^{M} \lambda_k^{n} \phi_k^{(v)}(x_j), \quad j = 1, 2, \cdots, M.
\end{aligned} \tag{15}$$

By systematically solving the system at each time step, we can elegantly construct the solution through Equation (12). This process is efficiently executed using MATLAB, allowing us to uncover insightful results with precision and ease. Let's now give the algorithm for the system (15).

Algorithm

- Choose the fractional order δ, final time T, spatial nodes N_x, and time levels N_t.
- Generate the spatial collocation points

  $$x_j, \qquad j = 1, 2, \cdots, N_x,$$

  and define the time step size

  $$\Delta t = \frac{T}{N_t}.$$

- Construct the Gaussian RBFs

  $$\phi_k(x) = \exp\left(-c(x - x_k)^2\right).$$

- Assemble the interpolation matrix

  $$A_{kj} = \phi_k(x_j).$$

- Compute the RBF derivative matrices such as

  $$D_1 = A_x A^{-1}, \qquad D_3 = A_{xxx} A^{-1}, \qquad D_5 = A_{xxxxx} A^{-1}.$$

- Initialize the numerical solution using the initial condition

  $$z^0 = z(x, 0).$$

- Calculate fractional terms and then solve nonlinear system for each time step by using Newton-Raphson method.
- Compute the exact solution, absolute errors.
- Plot the numerical solutions, exact solutions, and error surfaces.

4. Stability Analysis

At first, Equation (3) can be linearized by assuming the quantity z in the nonlinear term zz_x as locally constant q. Then, we get

$$D_t^{\delta} z(x,t) + \alpha q z_x(x,t) + \beta z_{xxx}(x,t) - \gamma z_{xxxxx}(x,t) = 0, \tag{16}$$

$$\begin{aligned} D_t^{\delta} z(x,t) &= \mu[z_j^{n+1} - z_j^n] \\ &+ \mu b_1 z_j^n + \mu \sum_{i=1}^{n-1} b_i^* z_j^{n-i} - \mu b_n z_j^0, \end{aligned} \tag{17}$$

where $b_i^* = b_{i+1} - b_i, \quad i = 1, 2, \cdots, n-1$. It can be expressed into the difference equation same with Equation (15), as

$$\begin{aligned} \mu \sum_{k=1}^{M} \lambda_k^{n+1} \phi_k(x_j) &+ \frac{q\alpha}{2} \sum_{k=1}^{M} \lambda_k^{n+1} \phi_k'(x_j) \\ &+ \frac{\beta}{2} \sum_{k=1}^{M} \lambda_k^{n+1} \phi_k'''(x_j) - \frac{\gamma}{2} \sum_{k=1}^{M} \lambda_k^{n+1} \phi_k^{v}(x_j) \\ &= -\mu \sum_{i=1}^{n} b_i^* \sum_{k=1}^{M} \lambda_k^{n-i} \phi_i(x_j) + \mu b_n \sum_{k=1}^{M} \lambda_k^0 \phi_k(x_j) - \mu b_1 \sum_{k=1}^{M} \lambda_k^n \phi_k(x_j) + \mu \sum_{k=1}^{M} \lambda_k^n \phi_k(x_j) \\ &- \frac{q\alpha}{2} \sum_{k=1}^{M} \lambda_k^n \phi_k'(x_j) - \frac{\beta}{2} \sum_{k=1}^{M} \lambda_k^n \phi_k'''(x_j) + \frac{\gamma}{2} \sum_{k=1}^{M} \lambda_k^n \phi_k^{v}(x_j), \quad j = 1, 2, \cdots, M. \end{aligned} \tag{18}$$

As in the von-Neumann stability analysis method, we assume that

$$\sum_{k=1}^{M} \lambda_k^n \phi_k(x_j) = \xi^n e^{i\theta x_j}, \tag{19}$$

in here $i^2 = -1$, the amplification factor is shown by ξ and θ is the wave number. If this equation is plugged into the linear difference equation, then equating the coefficients of $e^{i\theta x_j}$ gives this equality

$$\xi = \frac{-\mu \sum_{j=1}^{n} b_j^* \xi^{-j} + \mu b_n \xi^{-n} - \mu b_1 + \mu + \frac{i}{2}(-q\alpha\theta + \beta\theta^3 + \gamma\theta^5)}{\mu - \frac{i}{2}(-q\alpha\theta + \beta\theta^3 + \gamma\theta^5)}. \tag{20}$$

Let's consider the time independent limit value $\xi = -1$ as in [34], then

$$\mu \sum_{j=1}^{n} b_j^* \xi^{-j} + \mu b_n \xi^{-n} = -\mu(2^{1-\delta} - 1),$$

and

$$\mu b_1 = \mu(2^{1-\delta} - 1).$$

Equation (20) can be rewritten as

$$\xi = \frac{\mu + \frac{i}{2}(-q\alpha\theta + \beta\theta^3 + \gamma\theta^5)}{\mu - \frac{i}{2}(-q\alpha\theta + \beta\theta^3 + \gamma\theta^5)}. \tag{21}$$

In conclusion, we find that the method is unconditionally stable since $|\xi| = 1$.

5. Numerical Outcomes and Discussion

This part gives numerical results of the TFKE on the two test problem.

Test Problem 1:

For $\delta = 1$ and $f(x,t) = 0$, the exact solution of Equation (3) is

$$z(x,t) = \frac{105}{169}\text{sech}^4\left(\frac{1}{2\sqrt{13}}(x - \frac{36t}{169})\right),$$

with the initial condition is given as

$$z(x,0) = \frac{105}{169}\text{sech}^4\left(\frac{x}{2\sqrt{13}}\right). \tag{22}$$

Numerical calculation is done over the solution domain $-20 \le x \le 20$ in the time period $0 \le t \le 10$ with parameters $\Delta x = 0.05$ and $\Delta t = 0.1$. Numerical and exact solutions are presented in Figures 1, 3, 4, and 5 for the choice of $\delta = 1$. Additionaly, absolute errors are illustrated in Figures 2, and 6. All numerical results show that the method is effective and reliable.

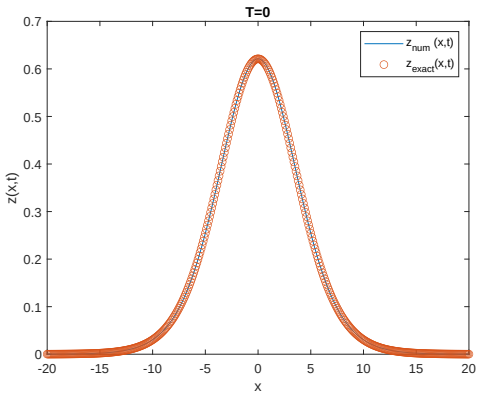


Figure 1: Numerical and exact solutions for $\delta = 1$

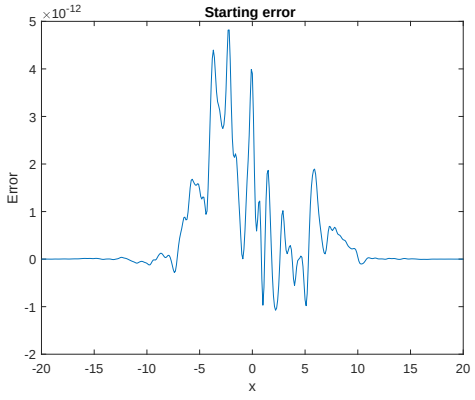


Figure 2: Starting error at $T = 0$ for Test Problem 1

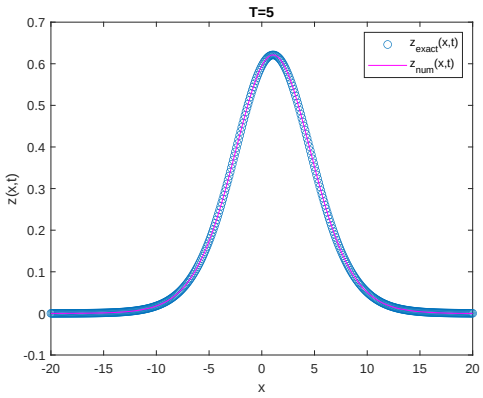


Figure 3: Numerical and exact solutions at $T = 5$ for Test Problem 1

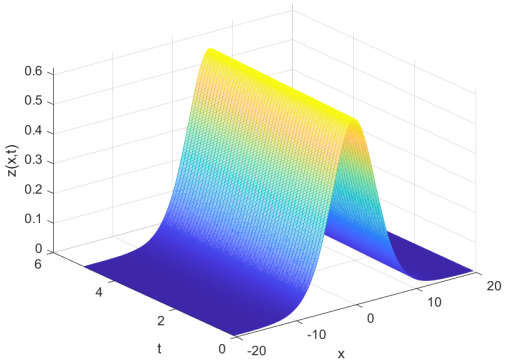


Figure 4: Numerical solution for $\delta = 1$, $\Delta x = 0.05$, and $\Delta t = 0.1$

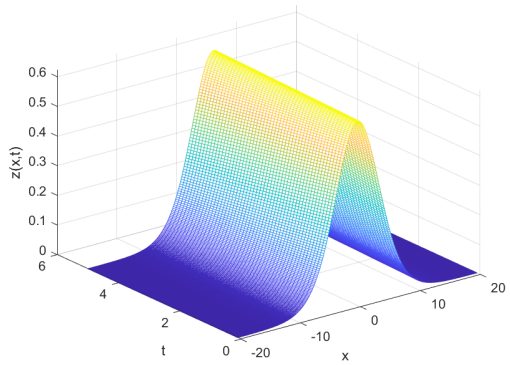


Figure 5: Exact solution for $\delta = 1$, $\Delta x = 0.05$, and $\Delta t = 0.1$

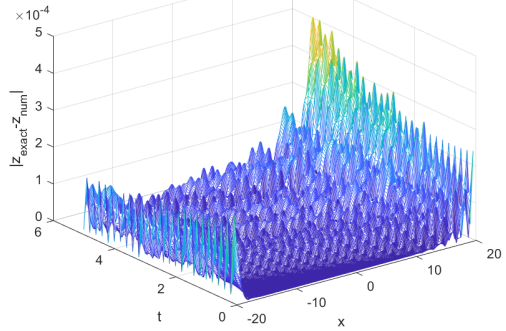


Figure 6: Absolute error for $\delta = 1$

Test Problem 2: To effectively validate the proposed numerical method, we employ a manufactured solution approach. In this process, we strategically select the exact solution as

$$z(x,t) = t^{2+\delta}\sin(\pi x),$$

and carefully construct the corresponding source term to guarantee that the equation is satisfied precisely. This recognized scheme serves as a strong means of verifying numerical techniques for fractional differential equations, ensuring reliability and accuracy. The forcing term $f(x,t)$ is also computed by substituting the exact solution into the equation.

The numerical simulations conducted using the RBF collocation method provide valuable insights into the effectiveness of this approach for approximating solutions to the TFKE across a range of fractional-order parameters δ. The strong agreement between the numerical and exact solutions throughout the computational domain reflects the method's high accuracy and stability.

The three-dimensional surface plots 7, 8, 9, 10 offer a clear illustration of how the solution evolves smoothly in both space and time. As the fractional order transitions from $\delta = 0.4$ to $\delta = 1.0$, we observe an increase in the amplitude of the solution and more pronounced wave propagation.

This behavior aligns with expectations, as a higher δ reduces the memory effect associated with the fractional derivative, guiding the model toward the classical Kawahara equation when $\delta = 1$.

The comparison plots at the final time level 11 reinforce the RBF collocation technique's ability to effectively capture the complex nonlinear and dispersive dynamics of the equation. The near overlap of the numerical curves with the exact solutions across all tested fractional orders highlights the algorithm's reliability and effectiveness.

As shown in Figure 12, the numerical error consistently decreases steadily as the numbers of collocation points and time levels increases. In summary, the results suggest that the RBF collocation method serves as a promising and accurate numerical framework for solving the time-fractional Kawahara equation across various fractional orders. The established error bounds further support its reliability, making it an appealing choice for practical applications in the field.

Table 1 shows the maximum absolute errors achieved by the RBF collocation method for different spatial and temporal discretizations. The results also show that the fully implicit scheme remains stable and efficient, even when dealing with nonlinear and fifth-order dispersive terms.

Ultimately, the Table 1 confirms that the proposed method demonstrates reliable numerical performance with minimal errors.

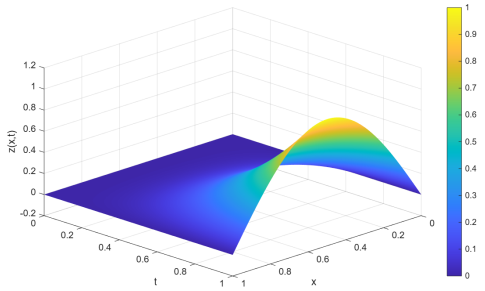


Figure 7: Numerical solution for $\delta = 0.4$

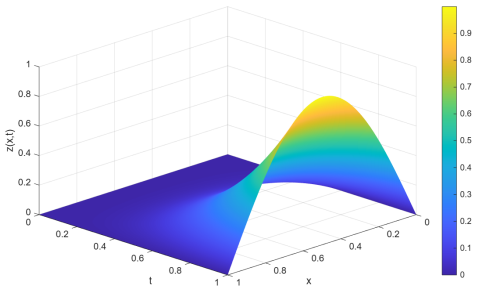


Figure 8: Numerical solution for $\delta = 0.6$

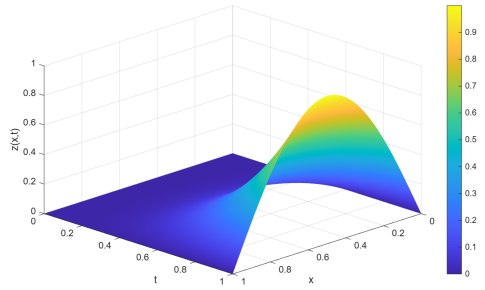


Figure 9: Numerical solution for $\delta = 0.8$

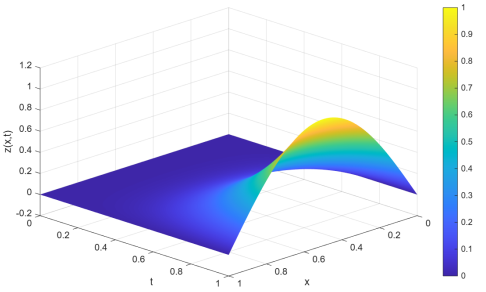


Figure 10: Numerical solution for $\delta = 1$

In here, N_x represents the number of spatial collocation points employed in the RBF approximation over the spatial domain, while N_t denotes the number of temporal discretization

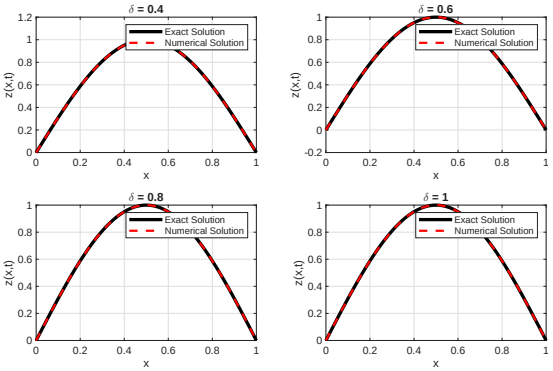


Figure 11: Comparison of numerical and exact solutions for $\delta = 0.4,\ 0.6,\ 0.8,$ and 1

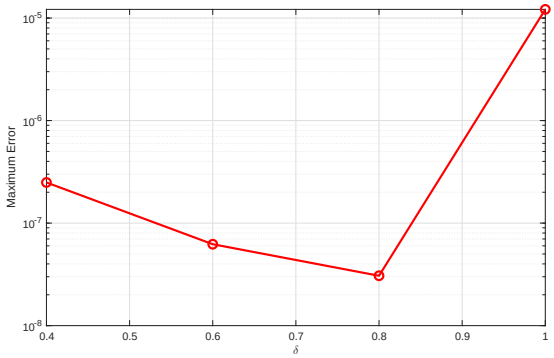


Figure 12: Maximum error for different values of δ

Table 1: Maximum absolute errors for the RBF collocation method applied to Test Problem 2

N_x	N_t	L_∞
21	300	$1.144899e-08$
41	600	$3.991269e-08$
61	900	$3.810005e-08$
81	1200	$2.840317e-08$

points utilized in the L1 finite difference approximation of the Liouville-Caputo FD.

Recent research utilizing spectral collocation methods for the nonlinear time-fractional generalized Kawahara equation has revealed impressive accuracy and robust convergence properties. Notably, eighth-kind Chebyshev collocation schemes [2] and operational matrix techniques have demonstrated the ability to provide highly precise approximations with minimal computational errors.

Our current findings are consistent with these advanced spectral methods, yet they also offer the advantage of simpler implementation through RBF interpolation. Moreover, unlike many conventional polynomial-based spectral methods, the RBF collocation approach eliminates the need for structured basis functions or complex orthogonal polynomial constructions, making it a more accessible and efficient choice for practitioners. The numerical error behavior observed in our simulations aligns closely with established theoretical analyses for fractional differential equations. Studies on spectral and finite difference methods for time-fractional models have shown that the L1 approximation typically achieves a temporal accuracy of $O(\Delta t^{2-\delta})$. This finding supports the convergence trend we've identified in our computations.

Additionally, prior research on local RBF-FD methods for KE has demonstrated that RBF-based discretizations maintain stability and accuracy, even when faced with equations containing fifth-order dispersive terms. Our current results reinforce the idea that RBF collocation is highly effective at approximating nonlinear fractional KE across various fractional orders.

In summary, the evidence clearly indicates that our proposed method stands out as competitive compared to existing spectral, finite difference, and operational matrix techniques, while

delivering:

- Exceptional spatial accuracy,
- Effortless implementation,
- Flexibility for nonlinear fractional PDEs,
- Consistent convergence behavior across diverse fractional orders.

Choosing our approach ensures you harness the full potential of modern numerical methods for complex problems.

6. Conclusion

This paper begins by demonstrating a mesh-free approximation using the RBF collocation method to numerically solve the TFKE. The time-fractional derivative is expressed by the Liouville-Caputo FD. We use two test problems to obtain the numerical solutions for the present equation.

In the first test problem, we compute numerical solution for the fractional derivative with a value of $\delta = 1$. Graphical results are given in figures.

In the second test problem, a well-designed manufactured test problem with an exact analytical solution is used to validate the effectiveness of the method. Numerical simulations are conducted for various fractional-order values, and the resulting numerical solutions display an excellent match with the corresponding exact solutions. The graphical outcomes indicate that the proposed method effectively captures the nonlinear and dispersive behavior of the equation while maintaining numerical stability throughout the computational domain.

The error analysis approves the method's convergence and reliability. Maximum absolute errors significantly decrease as the number of spatial collocation points and temporal discretization levels increases. Furthermore, the stability analysis of the approach is performed using a method akin to the traditional von-Neumann analysis. The consequence indicates that it is unconditionally stable.

Our main results display the idea that RBF collocation is highly effective at approximating nonlinear fractional Kawahara equations across various fractional orders. Thus, it can be said that the mentioned method provides an efficient alternative for solving these fractional PDEs due to its straightforward implementation.

Acknowledgement

This paper is supported by Eskişehir Technical University Research Fund under Contract No. 23ADP045.

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and legal special permission.

Authors Contributions

Author [Bahar Karaman]: Collected the data, contributed to completing the research and solving the problem, wrote the manuscript (%45).

Author [Emrah Karaman]: Contributed to completing the research and solving the problem (%25).

Author [Yılmaz Dereli]: Contributed to research method or evaluation of data, contributed to completing the research and solving the problem (%30).

Conflicts of Interest

The authors declare no conflict of interest.

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