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**Research Article**

# The $q$-Analogues of Min and Max Matrices and their Algebraic Properties

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**Abstract:** This study introduces a new family of matrices, the $q$-analogues, by generalizing the classical Min and Max matrices with entries from the $q$-calculus. The fundamental algebraic properties of these matrices, such as determinant, inverse, characteristic polynomial, $LU$ decomposition, and Frobenius and spectral norms, are systematically investigated. The results are supported by low-dimensional numerical examples demonstrating the validity of the formulas. This study makes a significant contribution to the literature on Min and Max matrices by providing a comprehensive analysis of a $q$-dependent parametric extension of these matrices.

**Keywords:** Min matrix, Max matrix, $q$-calculus, algebraic properties.

## 1. Introduction

The $q$-calculus, often called quantum calculus, is a generalization of classical calculus that eliminates the notation of limits and provides new ways to define derivatives and integrals. It includes a $q$-parameter to extend fundamental mathematical concepts. For example, the $q$-analogue of an integer $k$ is defined as

$$[k]_q = 1 + q + q^2 + \cdots + q^{k-1}$$

[15]. By convention, $[0]_q = 0$. In $q$-calculus, $q$-integers serve the role of ordinary integers in classical calculus. Many fundamental concepts, such as the $q$-factorial and the $q$-binomial coefficients, are based on $q$-integers. They are defined by

$$[k]_q! = [1]_q[2]_q\cdots[k]_q \quad \text{and} \quad \binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

[15]. Kızılateş et al. [20] introduced a new family of hybrid numbers defined via $q$-integers and higher-order generalized Fibonacci numbers and established their comprehensive algebraic

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structure, deriving a Binet-like formula, novel identities, generating functions, and summation properties. Yener and Emiroğlu [36] introduced the $q$-analogues of fundamental concepts in multiplicative calculus, establishing basic theorems for derivatives, integrals, and infinite products within this framework. This framework extends to $q$-derivatives and $q$-integrals, creating a rich language with deep applications in combinatorics, special functions, and quantum groups. In recent decades, this language has been productively imported into linear algebra, giving rise to a wide array of structured $q$-matrices. For instance, the $q$-analogue of the classical Pascal matrix was introduced by Zheng [37]. Tuğlu and Kuş [34] later defined $q$-Bernoulli and $q$-Bernoulli polynomial matrices, investigating their properties. Other notable examples include the $q$-Catalan matrix studied by Yaying et al. [35], and the $q$-distance and $q$-Laplacian matrices for trees, whose algebraic properties were analyzed by Bapat et al. [2] and later extended to weighted trees by Barik et al. [3]. Cheon et al. [5] generalized the concept of exponential Riordan matrices to a $q$-setting. A more systematic approach to $q$-deformed matrix computation was presented by Ernst [7], who subsequently developed related techniques for $q$-Appell, $q$-Bernoulli, and $q$-Euler polynomials [8] and established connections among $q$-Cauchy, $q$-Bernoulli, and $q$-Euler matrices [9]. Recently, Mersin [23] defined the $q$-derivative operator for matrix functions, introducing new structures such as the $q$-matrix exponential function, $q$-binomial of two matrices, $q$-trigonometric and $q$-hyperbolic matrix functions, and examined their fundamental properties in detail. Moreover, the $q$-analogue of the Frank matrix, which is a special type of Max matrix, was defined, and some of its properties are investigated, such as determinant, inverse, characteristic polynomials and eigenvalues in [24]. For another generalization of the Frank matrix, called the $r$-Frank matrix, see [30]. Further examples of $q$-analogues can be found in [18] on $(p,q)$-Chebyshev polynomials and in [19] on $q$-Fibonacci and $q$-Lucas octonions.

The success of $q$-calculus in generating structured matrices naturally leads to a question: can this powerful framework be applied to other fundamental matrix families, such as Min and Max matrices?

Min and Max matrices, whose entries are given by the minimum and maximum of their row and column indices, respectively, were introduced by Pólya and Szegő [29] as follows

$$A_{\min} = [\min(i,j)]_{i,j=1}^{n} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 \\ 1 & 2 & 3 & \cdots & 3 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & n-1 & n-1 \\ 1 & 2 & 3 & \cdots & n-1 & n \end{bmatrix}$$

and

$$A_{\max} = [\max(i,j)]_{i,j=1}^{n} = \begin{bmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 2 & 3 & \cdots & n-1 & n \\ 3 & 3 & 3 & \cdots & n-1 & n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n-1 & n-1 & n-1 & \cdots & n-1 & n \\ n & n & n & \cdots & n & n \end{bmatrix}.$$

Based on a real sequence $\{a_k\}$ with $a_1 < a_2 < \ldots < a_n$, the well-known generalizations $\mathcal{A}_{\min} = \left[a_{\min(i,j)}\right]_{i,j=1}^{n}$ and $\mathcal{A}_{\max} = \left[a_{\max(i,j)}\right]_{i,j=1}^{n}$ have been introduced, and their properties such as determinants, inverses, $LU$ decompositions, eigenvalues, and norms have been studied [6, 16, 21, 25, 26]. Petroudi and Pirouz [12, 13] studied Min and Max matrices whose entries are taken from the Fibonacci and Lucas sequences, and derived their properties in terms of these numbers. Similarly, the characteristic polynomials, determinants, and inverses of Min and Max matrices with entries from the hyper-Fibonacci and hyper-Lucas sequences were investigated in [27] and [32]. In addition, certain norms of Min matrices with harmonic number entries can be found in [14]. Some factorizations of generalized Min and Max matrices have been obtained by Kılıç and Arıkan [16], while the works of [33] and [21] also contain significant results related to Min and Max matrices. Another generalization, the $r$-Min and $r$-Max matrices, was introduced by Kızılateş and Terzioğlu [17], who examined their algebraic properties in detail. Further studies on the generalizations of $r$-Min and $r$-Max matrices, and hence Min and Max matrices, can be found in [1, 4, 10, 22, 28, 31].

Motivated by the diverse generalizations of Min and Max matrices and the established framework of $q$-calculus in matrix theory, we introduce their $q$-analogues. We derive explicit formulas for their determinants, inverses, characteristic polynomials, and $LU$ decompositions. Furthermore, we investigate their Frobenius and spectral norms, establishing upper bounds via Hadamard factorization. To ground our theoretical results, we include a numerical example for low dimensions. Our work systematically extends the classical theory of Min and Max matrices into the $q$-domain, offering a new family of structured matrices with explicitly computable properties. Although the general framework of $q$-analogues has been applied to other matrices such as the $q$-Frank matrix, the dense and symmetric structure of Min and Max matrices leads to significantly simpler algebraic properties, including tridiagonal inverses and elementary $LU$ decompositions. These features distinguish the $q$-Min and $q$-Max matrices from previously studied $q$-matrices such as the $q$-Frank matrix.

## 2. Preliminaries

The following definitions and results related to matrix norms and the Hadamard product can be found in [11].

Let $A \in \mathbb{C}^{m\times n}$ and let $A^H$ denote its conjugate transpose.

The spectral norm of $A$ is defined as

$$\|A\|_2 = \sqrt{\max_{1\le i\le n} \lambda_i(A^H A)},$$

where $\lambda_i(A^H A)$ are the eigenvalues of $A^H A$.

The Frobenius (Euclidean) norm of $A$ is defined as

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2}.$$

For any matrix $A$, the following inequality holds

$$\|A\|_2 \le \|A\|_F.$$

The maximum row length norm and maximum column length norm are given by

$$r_1(A) = \max_{1\le i\le m} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2}, \qquad c_1(A) = \max_{1\le j\le n} \sqrt{\sum_{i=1}^{m} |a_{ij}|^2}.$$

The following inequality is known

$$\|X \circ Y\|_2 \le r_1(X)\, c_1(Y),$$

where $X \circ Y$ denotes the Hadamard product of $X$ and $Y$, where $X, Y \in \mathbb{C}^{m\times n}$.

## 3. Main Results

**Definition 3.1** *The matrix* $A_{\min}^{(q)} = \left[[\min(i,j)]_q\right]_{i,j=1}^{n}$ *is called* $q$*-Min matrix whose explicit form is*

$$A_{\min}^{(q)} = \begin{bmatrix} [1]_q & [1]_q & [1]_q & \cdots & [1]_q & [1]_q \\ [1]_q & [2]_q & [2]_q & \cdots & [2]_q & [2]_q \\ [1]_q & [2]_q & [3]_q & \cdots & [3]_q & [3]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [1]_q & [2]_q & [3]_q & \cdots & [n-1]_q & [n-1]_q \\ [1]_q & [2]_q & [3]_q & \cdots & [n-1]_q & [n]_q \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1+q & 1+q & \cdots & 1+q & 1+q \\ 1 & 1+q & 1+q+q^2 & \cdots & 1+q+q^2 & 1+q+q^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1+q & 1+q+q^2 & \cdots & 1+q+q^2+\cdots+q^{n-2} & 1+q+q^2+\cdots+q^{n-2} \\ 1 & 1+q & 1+q+q^2 & \cdots & 1+q+q^2+\cdots+q^{n-2} & 1+q+q^2+\cdots+q^{n-1} \end{bmatrix}.$$

For $q \to 1$ the $q$-Min matrix $A_{\min}^{(q)}$ corresponds to the well-known Min matrix $A_{\min}$, whose entries are given by $[\min(i,j)]_{i,j=1}^{n}$.

**Theorem 3.2** *The determinant of the matrix* $A_{\min}^{(q)}$ *is calculated by the rule*

$$\det\left(A_{\min}^{(q)}\right) = \prod_{k=1}^{n-1} q^k.$$

**Proof** We perform the column operations $C_{i+1} \leftarrow C_{i+1} - C_i$ for $i = 1, 2, \ldots, n-1$, where $C_i$ denotes the $i$-th column of the matrix. Then, we have

$$\begin{aligned}
\det\left(A_{\min}^{(q)}\right) &= \begin{bmatrix}
[1]_q & 0 & 0 & \cdots & 0 & 0 \\
[1]_q & [2]_q - [1]_q & 0 & \cdots & 0 & 0 \\
[1]_q & [2]_q - [1]_q & [3]_q - [2]_q & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
[1]_q & [2]_q - [1]_q & [3]_q - [2]_q & \cdots & [n-1]_q - [n-2]_q & 0 \\
[1]_q & [2]_q - [1]_q & [3]_q - [2]_q & \cdots & [n-1]_q - [n-2]_q & [n]_q - [n-1]_q
\end{bmatrix} \\
&= [1]_q \left([2]_q - [1]_q\right)\left([3]_q - [2]_q\right) \cdots \left([n]_q - [n-1]_q\right) \\
&= \prod_{k=1}^{n-1} \left([k+1]_q - [k]_q\right) \\
&= \prod_{k=1}^{n-1} q^k.
\end{aligned}$$

□

**Theorem 3.3** *The inverse of the matrix* $A_{\min}^{(q)}$*, denoted by* $R^{(q)} = [r_{ij}]_{i,j=1}^{n}$*, has entries given by*

$$r_{ij} = \begin{cases}
1 + \dfrac{1}{q}, & i = j = 1 \\
-\dfrac{1}{q^i}, & j = i + 1 \\
-\dfrac{1}{q^j}, & i = j + 1 \\
\dfrac{1 + \dfrac{1}{q}}{q^{i-1}}, & i = j \neq 1, n \\
\dfrac{1}{q^{i-1}}, & i = j = n \\
[0]_q, & \textit{otherwise.}
\end{cases}$$

**Proof** The proof is done by induction on $n$. For $n = 2$, we have

$$A_{\min}^{(q)} = \begin{bmatrix} [1]_q & [1]_q \\ [1]_q & [2]_q \end{bmatrix},$$

whose inverse is

$$R_2^{(q)} = \begin{bmatrix} 1 + \dfrac{1}{q} & -\dfrac{1}{q} \\ -\dfrac{1}{q} & \dfrac{1}{q} \end{bmatrix}.$$

The hypothesis holds for $n = 2$. Assuming the hypothesis holds for $n - 1$, we show that it also holds for $n$. For this purpose, we partition the matrices $A_{\min}^{(q)}$ and $R^{(q)}$ into block matrices as

follows

$$A_{\min}^{(q)} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \quad \text{and} \quad R^{(q)} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix},$$

where

$$Q_{11} = \begin{bmatrix} [1]_q & [1]_q & [1]_q & \cdots & [1]_q & [1]_q \\ [1]_q & [2]_q & [2]_q & \cdots & [2]_q & [2]_q \\ [1]_q & [2]_q & [3]_q & \cdots & [3]_q & [3]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [1]_q & [2]_q & [3]_q & \cdots & [n-2]_q & [n-2]_q \\ [1]_q & [2]_q & [3]_q & \cdots & [n-2]_q & [n-1]_q \end{bmatrix},$$

$$Q_{12} = \begin{bmatrix} [1]_q & [2]_q & [3]_q & \cdots & [n-2]_q & [n-1]_q \end{bmatrix}^T,$$

$$Q_{21} = \begin{bmatrix} [1]_q & [2]_q & [3]_q & \cdots & [n-2]_q & [n-1]_q \end{bmatrix},$$

and

$$Q_{22} = \begin{bmatrix} [n]_q \end{bmatrix}.$$

Note that, according to the assumption, $Q_{22}^{-1}$ equals the matrix $R^{(q)}$ of order $n-1$. The equation

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

where $I$ and $0$ represent the appropriately sized identity and zero matrices, yields the following results

$$R_{22} = \left(Q_{22} - Q_{21}Q_{11}^{-1}Q_{12}\right)^{-1} = \begin{bmatrix} \dfrac{1}{q^{n-1}} \end{bmatrix},$$

$$R_{21} = -R_{22}Q_{21}Q_{11}^{-1} = \begin{bmatrix} [0]_q & [0]_q & [0]_q & \cdots & -\dfrac{1}{q^{n-1}} \end{bmatrix},$$

$$R_{12} = -Q_{11}^{-1}Q_{12}R_{22} = \begin{bmatrix} [0]_q & [0]_q & [0]_q & \cdots & -\dfrac{1}{q^{n-1}} \end{bmatrix}^T,$$

and

$$R_{11} = Q_{11}^{-1} - Q_{11}^{-1}Q_{12}R_{22}Q_{21}Q_{11}^{-1} = \begin{bmatrix} 1+\dfrac{1}{q} & -\dfrac{1}{q} & [0]_q & \cdots & [0]_q \\ -\dfrac{1}{q} & \dfrac{1+\dfrac{1}{q}}{q} & -\dfrac{1}{q^2} & \cdots & [0]_q \\ [0]_q & -\dfrac{1}{q^2} & \dfrac{1+\dfrac{1}{q}}{q^2} & \cdots & [0]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ [0]_q & [0]_q & [0]_q & \cdots & -\dfrac{1}{q^{n-2}} \\ [0]_q & [0]_q & [0]_q & \cdots & \dfrac{1+\dfrac{1}{q}}{q^{n-2}} \end{bmatrix}.$$

Hence, we obtain

$$R^{(q)} = \begin{bmatrix} 1+\frac{1}{q} & -\frac{1}{q} & [0]_q & \cdots & [0]_q & [0]_q \\ -\frac{1}{q} & \frac{1+\frac{1}{q}}{q} & -\frac{1}{q^2} & \cdots & [0]_q & [0]_q \\ [0]_q & -\frac{1}{q^2} & \frac{1+\frac{1}{q}}{q^2} & \cdots & [0]_q & [0]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [0]_q & [0]_q & [0]_q & \cdots & \frac{1+\frac{1}{q}}{q^{n-2}} & -\frac{1}{q^{n-1}} \\ [0]_q & [0]_q & [0]_q & \cdots & -\frac{1}{q^{n-1}} & \frac{1}{q^{n-1}} \end{bmatrix},$$

which completes the proof. □

**Theorem 3.4** *The characteristic polynomial of the matrix* $A_{\min}^{(q)}$ *satisfies the recurrence relation for* $n \geq 2$,

$$P_n(\lambda) = \left(q^{n-1} - 2\lambda\right)P_{n-1}(\lambda) - \lambda^2 P_{n-2}(\lambda)$$

*with initials* $P_0(\lambda) = 1$ *and* $P_1(\lambda) = 1 - \lambda$.

**Proof** To obtain the characteristic polynomial, we use the equality $P_n(\lambda) = |A_{\min}^{(q)} - \lambda I|$, which gives

$$P_n(\lambda) = \begin{vmatrix} [1]_q - \lambda & [1]_q & [1]_q & \cdots & [1]_q & [1]_q \\ [1]_q & [2]_q - \lambda & [2]_q & \cdots & [2]_q & [2]_q \\ [1]_q & [2]_q & [3]_q - \lambda & \cdots & [3]_q & [3]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [1]_q & [2]_q & [3]_q & \cdots & [n-1]_q - \lambda & [n-1]_q \\ [1]_q & [2]_q & [3]_q & \cdots & [n-1]_q & [n]_q - \lambda \end{vmatrix}.$$

To compute this determinant, for $i = 1, 2, \ldots, n-1$, we subtract the $i$th column from the $(i+1)$th column, and then subtract the $i$th row from the $(i+1)$th row. This produces the following determinant

$$P_n(\lambda) = \begin{vmatrix} [1]_q - \lambda & \lambda & 0 & \cdots & 0 & 0 \\ \lambda & [2]_q - [1]_q - 2\lambda & \lambda & \cdots & 0 & 0 \\ 0 & \lambda & [3]_q - [2]_q - 2\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & [n-1]_q - [n-2]_q - 2\lambda & \lambda \\ 0 & 0 & 0 & \cdots & \lambda & [n]_q - [n-1]_q - 2\lambda \end{vmatrix}.$$

By expanding along the last row, we obtain

$$P_n(\lambda) = \left([n]_q - [n-1]_q - 2\lambda\right)P_{n-1}(\lambda) - \lambda^2 P_{n-2}(\lambda),$$

which completes the proof. □

**Theorem 3.5** *The components of the LU decomposition of the matrix* $A_{\min}^{(q)}$ *are* $L = [l_{ij}]_{i,j=1}^{n}$ *and* $U = [u_{ij}]_{i,j=1}^{n}$, *whose entries are given as follows*

$$l_{ij} = \begin{cases} [0]_q, & i < j \\ [1]_q, & i \geq j \end{cases}$$

*and*

$$u_{ij} = \begin{cases} [0]_q, & i > j \\ q^{i-1}, & i \leq j. \end{cases}$$

**Proof** The proof is obtained directly from the matrix multiplication $A_{\min}^{(q)} = LU$. □

**Theorem 3.6** *For the matrix* $A_{\min}^{(q)}$, *we have the following results.*

*(i) The Frobenius norm satisfies the formula*

$$\|A_{\min}^{(q)}\|_F = \sqrt{\sum_{i=1}^{n} (2i-1)\,[n+1-i]_q^2},$$

*(ii) The spectral norm has the upper bound*

$$\|A_{\min}^{(q)}\|_2 \leq \sqrt{\left(\sum_{i=1}^{n} [i]_q^2\right)\left(\sum_{i=1}^{n} [i]_q^2 + 1\right)}.$$

**Proof**

(i) For the Frobenius norm of $A_{\min}^{(q)}$, we have

$$\begin{aligned} \|A_{\min}^{(q)}\|_F^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} [\min(i,j)]_q^2 \\ &= 1[n]_q^2 + 3[n-1]_q^2 + 5[n-2]_q^2 + \cdots + (2n-1)[1]_q^2 \\ &= \sum_{i=1}^{n} (2i-1)\,[n+1-i]_q^2. \end{aligned}$$

(ii) For the spectral norm of $A_{\min}^{(q)}$, we use the Hadamard product as

$$A_{\min}^{(q)} = A_1 \circ A_2,$$

where

$$A_1 = \begin{bmatrix} [1]_q & [1]_q & [1]_q & \cdots & [1]_q & [1]_q \\ [1]_q & [2]_q & [1]_q & \cdots & [1]_q & [1]_q \\ [1]_q & [2]_q & [3]_q & \cdots & [1]_q & [1]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [1]_q & [2]_q & [3]_q & \cdots & [n-1]_q & [1]_q \\ [1]_q & [2]_q & [3]_q & \cdots & [n-1]_q & [n]_q \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} [1]_q & [1]_q & [1]_q & \cdots & [1]_q & [1]_q \\ [1]_q & [1]_q & [2]_q & \cdots & [2]_q & [2]_q \\ [1]_q & [1]_q & [1]_q & \cdots & [3]_q & [3]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [1]_q & [1]_q & [1]_q & \cdots & [1]_q & [n-1]_q \\ [1]_q & [1]_q & [1]_q & \cdots & [1]_q & [1]_q \end{bmatrix}.$$

Then, we have

$$r_1(A_1) = \sqrt{\sum_{i=1}^{n} [i]_q^2} \quad \text{and} \quad c_1(A_2) = \sqrt{\sum_{i=1}^{n-1} [i]_q^2 + [1]_q^2}.$$

Hence, the inequality

$$\|A_{\min}^{(q)}\|_2 \leq r_1(A_1)\, c_1(A_2)$$

completes the proof.

□

**Definition 3.7** *The matrix*

$$A_{\max}^{(q)} = \left[ [\max(i,j)]_q \right]_{i,j=1}^{n}$$

*is called the* $q$-Max matrix. *Its explicit form can be expressed as*

$$A_{\max}^{(q)} = \begin{bmatrix} [1]_q & [2]_q & [3]_q & \cdots & [n-1]_q & [n]_q \\ [2]_q & [2]_q & [3]_q & \cdots & [n-1]_q & [n]_q \\ [3]_q & [3]_q & [3]_q & \cdots & [n-1]_q & [n]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [n-1]_q & [n-1]_q & [n-1]_q & \cdots & [n-1]_q & [n]_q \\ [n]_q & [n]_q & [n]_q & \cdots & [n]_q & [n]_q \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1+q & 1+q+q^2 & \cdots & 1+q+\cdots+q^{n-2} & 1+q+\cdots+q^{n-1} \\ 1+q & 1+q & 1+q+q^2 & \cdots & 1+q+\cdots+q^{n-2} & 1+q+\cdots+q^{n-1} \\ 1+q+q^2 & 1+q+q^2 & 1+q+q^2 & \cdots & 1+q+\cdots+q^{n-2} & 1+q+\cdots+q^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1+q+\cdots+q^{n-2} & 1+q+\cdots+q^{n-2} & 1+q+\cdots+q^{n-2} & \cdots & 1+q+\cdots+q^{n-2} & 1+q+\cdots+q^{n-1} \\ 1+q+\cdots+q^{n-1} & 1+q+\cdots+q^{n-1} & 1+q+\cdots+q^{n-1} & \cdots & 1+q+\cdots+q^{n-1} & 1+q+\cdots+q^{n-1} \end{bmatrix}.$$

As $q \to 1$, the $q$-Max matrix $A_{\max}^{(q)}$ reduces to the well-known Max matrix $A_{\max}$, whose entries are given by $[\max(i,j)]_{i,j=1}^{n}$.

**Theorem 3.8** *The determinant of* $A_{\max}^{(q)}$ *is given by*

$$\det\left(A_{\max}^{(q)}\right) = [n]_q \prod_{k=1}^{n-1} -q^k.$$

**Proof** We perform the row operations $R_i \leftarrow R_i - R_{i+1}$ for $i = 1, 2, \ldots, n-1$, where $R_i$ denotes the $i$-th row of the matrix. Then, we obtain

$$\det\left(A_{\max}^{(q)}\right) = \begin{bmatrix} [1]_q - [2]_q & 0 & 0 & \cdots & 0 & 0 \\ [2]_q - [3]_q & [2]_q - [3]_q & 0 & \cdots & 0 & 0 \\ [3]_q - [4]_q & [3]_q - [4]_q & [3]_q - [4]_q & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [n-1]_q - [n]_q & [n-1]_q - [n]_q & [n-1]_q - [n]_q & \cdots & [n-1]_q - [n]_q & 0 \\ [n]_q & [n]_q & [n]_q & \cdots & [n]_q & [n]_q \end{bmatrix}$$

$$= [n]_q \prod_{k=1}^{n-1} ([k]_q - [k+1]_q)$$
$$= [n]_q \prod_{k=1}^{n-1} -q^k.$$

□

**Theorem 3.9** *The inverse of the matrix $A_{\max}^{(q)}$, denoted by $S^{(q)} = [s_{ij}]$, has entries characterized as follows*

$$s_{ij} = \begin{cases} -\dfrac{1}{q}, & i = j = 1 \\ \dfrac{1}{q^i}, & j = i + 1 \\ \dfrac{1}{q^j}, & i = j + 1 \\ 1 + \dfrac{1}{q}, & \\ -\dfrac{q}{q^{i-1}}, & i = j \neq 1, n \\ -\dfrac{[i-1]_q}{q^{i-1}[i]_q}, & i = j = n \\ [0]_q & \textit{otherwise.} \end{cases}$$

**Proof** The proof proceeds by induction on $n$. In the base case $n = 2$, the theorem is satisfied, as

$$A_{\max}^{(q)} = \begin{bmatrix} [1]_q & [2]_q \\ [2]_q & [2]_q \end{bmatrix} \quad \text{and} \quad S_2^{(q)} = \begin{bmatrix} -\dfrac{1}{q} & \dfrac{1}{q} \\ \dfrac{1}{q} & -\dfrac{1}{q(q+1)} \end{bmatrix}.$$

Assume the statement holds for $n-1$. To prove it for $n$, we apply the induction hypothesis to the leading principal submatrix of order $n-1$. For this purpose, we partition the matrices as follows

$$A_{\max}^{(q)} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \quad \text{and} \quad S^{(q)} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix},$$

where

$$U_{11} = \begin{bmatrix} [1]_q & [2]_q & [3]_q & \cdots & [n-2]_q & [n-1]_q \\ [2]_q & [2]_q & [3]_q & \cdots & [n-2]_q & [n-1]_q \\ [3]_q & [3]_q & [3]_q & \cdots & [n-2]_q & [n-1]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [n-2]_q & [n-2]_q & [n-2]_q & \cdots & [n-2]_q & [n-1]_q \\ [n-1]_q & [n-1]_q & [n-1]_q & \cdots & [n-1]_q & [n-1]_q \end{bmatrix},$$

$$U_{12} = \begin{bmatrix} [n]_q & [n]_q & [n]_q & \cdots & [n]_q & [n]_q \end{bmatrix}^T,$$

$$U_{21} = \begin{bmatrix} [n]_q & [n]_q & [n]_q & \cdots & [n]_q & [n]_q \end{bmatrix},$$

and

$$U_{22} = \begin{bmatrix} [n]_q \end{bmatrix}.$$

According to assumption, $U_{11}^{-1}$ corresponds to the matrix $S^{(q)}$ of order $n-1$. For appropriately sized identity matrix $I$ and zero matrix $0$, the equation

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

yields the block matrices of $S^{(q)}$ as follows

$$S_{22} = \left(U_{22} - U_{21}U_{11}^{-1}U_{12}\right)^{-1} = \left[ -\frac{[n-1]_q}{q^{n-1}[n]_q} \right],$$

$$S_{21} = -S_{22}U_{21}U_{11}^{-1} = \left[ [0]_q \quad [0]_q \quad [0]_q \quad \cdots \quad \frac{1}{q^{n-1}} \right],$$

$$S_{12} = -U_{11}^{-1}U_{12}S_{22} = \left[ [0]_q \quad [0]_q \quad [0]_q \quad \cdots \quad \frac{1}{q^{n-1}} \right]^T,$$

and

$$S_{11} = U_{11}^{-1} - U_{11}^{-1}U_{12}S_{22}U_{21}U_{11}^{-1} = \begin{bmatrix} -\frac{1}{q} & \frac{1}{q} & [0]_q & \cdots & [0]_q \\ \frac{1}{q} & -\frac{1+\frac{1}{q}}{q} & \frac{1}{q^2} & \cdots & [0]_q \\ [0]_q & \frac{1}{q^2} & -\frac{1+\frac{1}{q}}{q^2} & \cdots & [0]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ [0]_q & [0]_q & [0]_q & \cdots & \frac{1}{q^{n-2}} \\ [0]_q & [0]_q & [0]_q & \cdots & -\frac{1+\frac{1}{q}}{q^{n-2}} \end{bmatrix}.$$

Hence, we obtain

$$S^{(q)} = \begin{bmatrix} -\frac{1}{q} & \frac{1}{q} & [0]_q & \cdots & [0]_q & [0]_q \\ \frac{1}{q} & -\frac{1+\frac{1}{q}}{q} & \frac{1}{q^2} & \cdots & [0]_q & [0]_q \\ [0]_q & \frac{1}{q^2} & -\frac{1+\frac{1}{q}}{q^2} & \cdots & [0]_q & [0]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [0]_q & [0]_q & [0]_q & \cdots & -\frac{1+\frac{1}{q}}{q^{n-2}} & \frac{1}{q^{n-1}} \\ [0]_q & [0]_q & [0]_q & \cdots & \frac{1}{q^{n-1}} & -\frac{[n-1]_q}{q^{n-1}[n]_q} \end{bmatrix}.$$

□

**Theorem 3.10** *For the characteristic polynomial of the matrix $A_{\max}^{(q)}$, the following recurrence relation holds*

$$\psi_n(\mu) = ([n]_q - \mu)\,\phi_{n-1}(\mu) - \mu^2\phi_{n-2}(\mu),$$

*where*

$$\phi_{n-1}(\mu) = (-q^{n-1} - 2\mu)\phi_{n-2}(\mu) - \mu^2\phi_{n-3}(\mu)$$

*with initials $\phi_0(\mu) = 1$ and $\phi_1(\mu) = -q - 2\mu$.*

**Proof** Using the equality $\psi_n(\mu) = |A_{\max}^{(q)} - \mu I|$, we have

$$\psi_n(\mu) = \begin{vmatrix} [1]_q - \mu & [2]_q & [3]_q & \cdots & [n-1]_q & [n]_q \\ [2]_q & [2]_q - \mu & [3]_q & \cdots & [n-1]_q & [n]_q \\ [3]_q & [3]_q & [3]_q - \mu & \cdots & [n-1]_q & [n]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [n-1]_q & [n-1]_q & [n-1]_q & \cdots & [n-1]_q - \mu & [n]_q \\ [n]_q & [n]_q & [n]_q & \cdots & [n-1]_q & [n]_q - \mu \end{vmatrix}.$$

For $i = 1, 2, \ldots, n-1$, performing $R_i \leftarrow R_i - R_{i+1}$ and $C_i \leftarrow C_i - C_{i+1}$ yields

$$\psi_n(\mu) = \begin{vmatrix} [1]_q - [2]_q - 2\mu & \mu & 0 & \cdots & 0 & 0 \\ \mu & [2]_q - [3]_q - 2\mu & \mu & \cdots & 0 & 0 \\ 0 & \mu & [3]_q - [4]_q - 2\mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & [n-1]_q - [n]_q - 2\mu & \mu \\ 0 & 0 & 0 & \cdots & \mu & [n]_q - \mu \end{vmatrix}.$$

We now expand the determinant along the last row, and then we obtain

$$\psi_n(\mu) = ([n]_q - \mu)\begin{vmatrix} [1]_q-[2]_q-2\mu & \mu & 0 & \cdots & 0 & 0 \\ \mu & [2]_q-[3]_q-2\mu & \mu & \cdots & 0 & 0 \\ 0 & \mu & [3]_q-[4]_q-2\mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & [n-2]_q-[n-1]_q-2\mu & \mu \\ 0 & 0 & 0 & \cdots & \mu & [n-1]_q-[n]_q-2\mu \end{vmatrix}$$
$$-\mu\begin{vmatrix} [1]_q-[2]_q-2\mu & \mu & 0 & \cdots & 0 & 0 \\ \mu & [2]_q-[3]_q-2\mu & \mu & \cdots & 0 & 0 \\ 0 & \mu & [3]_q-[4]_q-2\mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & [n-2]_q-[n-1]_q-2\mu & 0 \\ 0 & 0 & 0 & \cdots & \mu & \mu \end{vmatrix}.$$

In the resulting expression, let the first determinant on the right-hand side be denoted by $\phi_{n-1}(\mu)$ and the second one by $X(\mu)$. When $\phi_{n-1}(\mu)$ is expanded along the last row, we obtain

$$\phi_{n-1}(\mu) =$$
$$([n-1]_q - [n]_q - 2\mu)\begin{vmatrix} [1]_q-[2]_q-2\mu & \mu & 0 & \cdots & 0 & 0 \\ \mu & [2]_q-[3]_q-2\mu & \mu & \cdots & 0 & 0 \\ 0 & \mu & [3]_q-[4]_q-2\mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & [n-3]_q-[n-2]_q-2\mu & \mu \\ 0 & 0 & 0 & \cdots & \mu & [n-2]_q-[n-1]_q-2\mu \end{vmatrix}$$
$$-\mu\begin{vmatrix} [1]_q-[2]_q-2\mu & \mu & 0 & \cdots & 0 & 0 \\ \mu & [2]_q-[3]_q-2\mu & \mu & \cdots & 0 & 0 \\ 0 & \mu & [3]_q-[4]_q-2\mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & [n-3]_q-[n-2]_q-2\mu & 0 \\ 0 & 0 & 0 & \cdots & \mu & \mu \end{vmatrix}.$$

In this case, the first determinant on the right-hand side is evidently $\phi_{n-2}(\mu)$. Expanding the second determinant along the last column yields

$$\begin{aligned} \phi_{n-1}(\mu) &= ([n-1]_q - [n]_q - 2\mu)\,\phi_{n-2}(\mu) - \mu^2\phi_{n-3}(\mu) \\ &= (-q^{n-1} - 2\mu)\phi_{n-2}(\mu) - \mu^2\phi_{n-3}(\mu) \end{aligned}$$

with initial conditions $\phi_0(\mu) = 1$ and $\phi_1(\mu) = -q - 2\mu$. Moreover, expanding $X(\mu)$ along the last column shows that it equals $\phi_{n-2}(\mu)$. Hence,

$$\psi_n(\mu) = ([n]_q - \mu)\,\phi_{n-1}(\mu) - \mu^2\phi_{n-2}(\mu).$$

□

**Theorem 3.11** *For the matrix $A^{(q)}_{\max}$, let its LU decomposition be given by $L = [l^*_{ij}]^n_{i,j=1}$ and $U = [u^*_{ij}]^n_{i,j=1}$. The entries of these matrices are characterized as follows*

$$l^*_{ij} = \begin{cases} [1]_q, & i = j = 1 \\ [0]_q, & i < j \\ \dfrac{[i]_q}{[j]_q}, & i > j \end{cases}$$

*and*

$$u^*_{ij} = \begin{cases} [j]_q, & i = 1 \\ -\dfrac{[i]_q q^{i-1}}{[i-1]_q}, & i = j \neq 1 \\ -\dfrac{[j]_q q^{i-1}}{[i-1]_q}, & i < j \\ [0]_q, & i > j. \end{cases}$$

**Proof** This result is obtained by verifying the equality $A_{\max}^{(q)} = LU$ through matrix multiplication. □

**Theorem 3.12** *For the matrix $A_{\max}^{(q)}$, the following hold*

*(i) The Frobenius norm satisfies the formula*

$$\|A_{\max}^{(q)}\|_F = \sqrt{\sum_{i=1}^{n}(2i-1)\,[i]_q^2},$$

*(ii) The spectral norm admits the upper bound*

$$\|A_{\max}^{(q)}\|_2 \leq \sqrt{n[n]_q^2\left((n-1)[n]_q^2+1\right)}.$$

**Proof**

(i) For the Frobenius norm of the matrix $A_{\max}^{(q)}$, we obtain

$$\begin{aligned}\|A_{\max}^{(q)}\|_F^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}[\max(i,j)]_q^2 \\ &= 1[1]_q^2+3[2]_q^2+5[3]_q^2+\cdots+(2n-1)[n]_q^2 \\ &= \sum_{i=1}^{n}(2i-1)\,[i]_q^2.\end{aligned}$$

(ii) To obtain the upper bound for the spectral norm of the matrix $A_{\max}^{(q)}$, we consider the Hadamard product

$$A_{\max}^{(q)} = A_1^* \circ A_2^*,$$

where

$$A_1^* = \begin{bmatrix} [1]_q & [1]_q & [1]_q & \cdots & [1]_q & [1]_q \\ [2]_q & [2]_q & [1]_q & \cdots & [1]_q & [1]_q \\ [3]_q & [3]_q & [3]_q & \cdots & [1]_q & [1]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [n-1]_q & [n-1]_q & [n-1]_q & \cdots & [n-1]_q & [1]_q \\ [n]_q & [n]_q & [n]_q & \cdots & [n]_q & [n]_q \end{bmatrix}$$

and

$$A_2^* = \begin{bmatrix} [1]_q & [2]_q & [3]_q & \cdots & [n-1]_q & [n]_q \\ [1]_q & [1]_q & [3]_q & \cdots & [n-1]_q & [n]_q \\ [1]_q & [1]_q & [1]_q & \cdots & [n-1]_q & [n]_q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [1]_q & [1]_q & [1]_q & \cdots & [1]_q & [n]_q \\ [1]_q & [1]_q & [1]_q & \cdots & [1]_q & [1]_q \end{bmatrix}.$$

Then, we obtain

$$r_1(A_1^*) = \sqrt{n[n]_q^2}, \qquad c_1(A_2^*) = \sqrt{(n-1)[n]_q^2+1}.$$

It follows that

$$\|A_{\max}^{(q)}\|_2 \leq r_1(A_1^*)\, c_1(A_2^*),$$

which completes the proof.

□

Next, we present an illustrative example.

**Example 3.13** *For $n = 5$ and $q = 2$, the matrices $A_{\min}^{(q)}$ and $A_{\max}^{(q)}$ are as follows*

$$A_{\min}^{(2)} = \begin{bmatrix} [1]_2 & [1]_2 & [1]_2 & [1]_2 & [1]_2 \\ [1]_2 & [2]_2 & [2]_2 & [2]_2 & [2]_2 \\ [1]_2 & [2]_2 & [3]_2 & [3]_2 & [3]_2 \\ [1]_2 & [2]_2 & [3]_2 & [4]_2 & [4]_2 \\ [1]_2 & [2]_2 & [3]_2 & [4]_2 & [5]_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 & 3 \\ 1 & 3 & 7 & 7 & 7 \\ 1 & 3 & 7 & 15 & 15 \\ 1 & 3 & 7 & 15 & 31 \end{bmatrix}$$

*and*

$$A_{\max}^{(2)} = \begin{bmatrix} [1]_2 & [2]_2 & [3]_2 & [4]_2 & [5]_2 \\ [2]_2 & [2]_2 & [3]_2 & [4]_2 & [5]_2 \\ [3]_2 & [3]_2 & [3]_2 & [4]_2 & [5]_2 \\ [4]_2 & [4]_2 & [4]_2 & [4]_2 & [5]_2 \\ [5]_2 & [5]_2 & [5]_2 & [5]_2 & [5]_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 7 & 15 & 31 \\ 3 & 3 & 7 & 15 & 31 \\ 7 & 7 & 7 & 15 & 31 \\ 15 & 15 & 15 & 15 & 31 \\ 31 & 31 & 31 & 31 & 31 \end{bmatrix}.$$

*Their determinants are*

$$\det\left(A_{\min}^{(2)}\right) = \prod_{k=1}^{4} 2^k = 1024$$

*and*

$$\det\left(A_{\max}^{(2)}\right) = [5]_q \prod_{k=1}^{4} -2^k = 31744.$$

*Their inverse matrices are*

$$\left(A_{\min}^{(2)}\right)^{-1} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{3}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{4} & \frac{3}{8} & -\frac{1}{8} & 0 \\ 0 & 0 & -\frac{1}{8} & \frac{3}{16} & -\frac{1}{16} \\ 0 & 0 & 0 & -\frac{1}{16} & \frac{1}{16} \end{bmatrix}$$

*and*

$$\left(A_{\max}^{(2)}\right)^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{3}{4} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{3}{8} & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{8} & -\frac{3}{16} & \frac{1}{16} \\ 0 & 0 & 0 & \frac{1}{16} & -\frac{15}{496} \end{bmatrix}.$$

*The characteristic polynomials of the matrices* $A^{(2)}_{\min}$ *for* $2 \le n \le 5$ *are*

$$\begin{aligned} P_2(\lambda) &= \lambda^2 - 4\lambda + 2, \\ P_3(\lambda) &= -\lambda^3 + 11\lambda^2 - 20\lambda + 8, \\ P_4(\lambda) &= \lambda^4 - 26\lambda^3 + 126\lambda^2 - 176\lambda + 64, \\ P_5(\lambda) &= -\lambda^5 + 57\lambda^4 - 648\lambda^3 + 2360\lambda^2 - 2944\lambda + 1024. \end{aligned}$$

*The characteristic polynomials of the matrices* $A^{(2)}_{\max}$ *for* $2 \le n \le 5$ *are*

$$\begin{aligned} \psi_2(\mu) &= \mu^2 - 4\mu - 6, \\ \psi_3(\mu) &= -\mu^3 + 11\mu^2 + 76\mu + 56, \\ \psi_4(\mu) &= \mu^4 - 26\mu^3 - 586\mu^2 - 1616\mu - 960, \\ \psi_5(\mu) &= -\mu^5 + 57\mu^4 + 3624\mu^3 + 27656, \mu^2 + 58496\mu + 31744, \end{aligned}$$

*where*

$$\begin{aligned} \phi_1(\mu) &= -2 - 2\mu, \\ \phi_2(\mu) &= 3\mu^2 + 12\mu + 8, \\ \phi_3(\mu) &= -4\mu^3 - 46\mu^2 - 112\mu - 64, \\ \phi_4(\mu) &= 5\mu^4 + 144\mu^3 + 952\mu^2 + 1920\mu + 1024. \end{aligned}$$

*The LU decompositions of the matrices* $A^{(2)}_{\min}$ *and* $A^{(2)}_{\max}$ *are*

$$A^{(2)}_{\min} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 4 & 4 & 4 \\ 0 & 0 & 0 & 8 & 8 \\ 0 & 0 & 0 & 0 & 16 \end{bmatrix}$$

*and*

$$A^{(2)}_{\max} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 7 & \frac{7}{3} & 1 & 0 & 0 \\ 15 & 5 & \frac{15}{7} & 1 & 0 \\ 31 & \frac{31}{3} & \frac{31}{7} & \frac{31}{15} & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 & 15 & 31 \\ 0 & -6 & -14 & -30 & -62 \\ 0 & 0 & \frac{-28}{3} & -20 & \frac{-124}{3} \\ 0 & 0 & 0 & \frac{-120}{7} & \frac{-248}{7} \\ 0 & 0 & 0 & 0 & \frac{-496}{15} \end{bmatrix}.$$

*The Frobenius norms of these matrices are*

$$\|A^{(2)}_{\min}\|_F = \sqrt{\sum_{i=1}^{5} (2i-1)\,[6-i]_2^2} = 44.19$$

*and*

$$\|A^{(2)}_{\max}\|_F = \sqrt{\sum_{i=1}^{5} (2i-1)\,[i]_2^2} = 102.45.$$

*The spectral norms of the matrices* $A^{(2)}_{\min}$ *and* $A^{(2)}_{\max}$ *are*

$$\|A^{(2)}_{\min}\|_2 = 43.24$$

*and*

$$\|A_{\max}^{(2)}\|_2 = 97.25,$$

*computed numerically using Maple. Now, we present the upper bounds for these spectral norms.*

*The matrices* $A_1$ *and* $A_2$ *according to Theorem 3.6 (ii) are*

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 3 & 7 & 1 & 1 \\ 1 & 3 & 7 & 15 & 1 \\ 1 & 3 & 7 & 15 & 31 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 3 & 3 \\ 1 & 1 & 1 & 7 & 7 \\ 1 & 1 & 1 & 1 & 15 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

*Then*

$$r_1(A_1) = \sqrt{\sum_{i=1}^{5} [i]_2^2} = \sqrt{1245}$$

*and*

$$c_1(A_2) = \sqrt{\sum_{i=1}^{4} [i]_2^2 + 1} = \sqrt{285}.$$

*Hence,*

$$\|A_{\min}^{(2)}\|_2 \le r_1(A_1)\, c_1(A_2) \approx 595.67.$$

*Also,*

$$\|A_{\min}^{(2)}\|_2 \le \|A_{\min}^{(2)}\|_F = 44.19.$$

*The matrices* $A_1^*$ *and* $A_2^*$ *in Theorem 3.12 (ii) are*

$$A_1^* = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 1 & 1 & 1 \\ 7 & 7 & 7 & 1 & 1 \\ 15 & 15 & 15 & 15 & 1 \\ 31 & 31 & 31 & 31 & 31 \end{bmatrix}, \qquad A_2^* = \begin{bmatrix} 1 & 3 & 7 & 15 & 31 \\ 1 & 1 & 7 & 15 & 31 \\ 1 & 1 & 1 & 15 & 31 \\ 1 & 1 & 1 & 1 & 31 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

*Then*

$$r_1(A_1^*) = \sqrt{5[5]_2^2} = \sqrt{4805}$$

*and*

$$c_1(A_2^*) = \sqrt{4[5]_2^2 + 1} = \sqrt{3845}.$$

*Hence,*

$$\|A_{\max}^{(2)}\|_2 \le r_1(A_1^*)\, c_1(A_2^*) \approx 4298.28.$$

*In addition,*

$$\|A_{\max}^{(2)}\|_2 \le \|A_{\max}^{(2)}\|_F = 102.45.$$

*The numerical results for the spectral norms of* $A_{\min}^{(2)}$ *and* $A_{\max}^{(2)}$ *for* $n = 2, 3, 4, 5$ *are summarised in Tables 1 and 2, respectively. The actual spectral norms and the Frobenius norms were computed numerically using Maple.*

Table 1: Actual spectral norms and upper bounds for $\|A_{\min}^{(2)}\|_2$

| $n$ | $\|A_{\min}^{(2)}\|_2$ (actual) | Upper bound (Theorem 3.6) | $\|A_{\min}^{(2)}\|_F$ |
|---|---|---|---|
| 2 | 3.41 | 4.47 | 3.46 |
| 3 | 8.84 | 25.48 | 9.00 |
| 4 | 20.18 | 130.54 | 20.59 |
| 5 | 43.24 | 595.67 | 44.19 |

Table 2: Actual spectral norms and upper bounds for $\|A_{\max}^{(2)}\|_2$

| $n$ | $\|A_{\max}^{(2)}\|_2$ (actual) | Upper bound (Theorem 3.12) | $\|A_{\max}^{(2)}\|_F$ |
|---|---|---|---|
| 2 | 5.16 | 13.42 | 5.29 |
| 3 | 15.98 | 120.64 | 16.52 |
| 4 | 41.19 | 780.00 | 42.99 |
| 5 | 97.25 | 4298.28 | 102.45 |

*The upper bounds in Tables 1 and 2, obtained from Theorems 3.6 and 3.12 respectively, are very loose for the spectral norm, especially for larger* $n$*. In contrast, the Frobenius norm provides a much sharper estimate, almost equal to the actual spectral norm. Thus, the Frobenius norm is a much better practical upper bound. The numerical values also show that the upper bounds grow rapidly with* $n$*, becoming increasingly loose.*

Now, we compare the $q$-Max matrix introduced in this paper with the $q$-Frank matrix, which is a special type of Max matrix studied in [24].

### 3.1. Comparison with the $q$-Frank Matrix

The $q$-Frank matrix defined in [24] has a lower Hessenberg structure, while the $q$-Max matrix introduced in this paper is dense and symmetric. Their determinants are $\prod_{k=1}^{n} q^{k-1}$ and $[n]_q \prod_{k=1}^{n-1}(-q^k)$, respectively. The inverse of the $q$-Frank matrix is upper Hessenberg, whereas the inverse of the $q$-Max matrix is tridiagonal. Both matrices satisfy second-order recurrence relations for their characteristic polynomials, but their explicit forms are different. These differences show that the present work is not a mere repetition but a natural extension to a different family of $q$-matrices.

## 4. Conclusion

In this paper, we have introduced the $q$-analogues of Min and Max matrices by defining their entries as $[\min(i,j)]_q$ and $[\max(i,j)]_q$, the $q$-integers. Thereby, we have embedded these well-studied matrix families into the richer framework of $q$-calculus. We have obtained explicit formulas

for the determinants, inverses, characteristic polynomials, $LU$ decompositions, Frobenius norms, and spectral norm bounds of both matrices. A numerical example confirms the validity of the formulas.

The results show that the $q$-analogues preserve the structural simplicity of the classical Min and Max matrices, such as tridiagonal inverses and elementary $LU$ factors, while introducing a $q$-dependence that allows further flexibility. This demonstrates that Min and Max structure leads to simpler algebraic properties compared to other $q$-matrices such as the $q$-Frank matrix, making the $q$-Min and $q$-Max matrices potentially useful in numerical linear algebra.

Future work includes a detailed spectral analysis and the investigation of sharper norm bounds for these matrices.

**Declaration of Ethical Standards**

The author declares that the materials and methods used in her study do not require ethical committee and legal special permission.

**Conflicts of Interest**

The author declares no conflict of interest.

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