

Convergence Results for Mappings Satisfying the (HRSC)-Condition in Uniformly Convex Hyperbolic Spaces

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Abstract

In this paper, we investigate fixed point approximation for mappings satisfying the Hardy-Rogers-Suzuki-type condition in uniformly convex hyperbolic spaces. By using a modified M -iteration process, we establish Δ -convergence and strong convergence results under suitable assumptions. An example illustrating the applicability of the obtained results is presented. The results extend and complement several existing convergence theorems for generalized nonexpansive mappings

Keywords: Fixed point, (HRSC)-condition, Generalized nonexpansive mapping, Hyperbolic space, Δ -convergence, Strong convergence.

1. Introduction

Fixed-point theory has become an important research area in nonlinear analysis, differential equations, optimization, and many applied sciences. The Banach contraction principle (Banach 1922) is considered one of the fundamental results in this area. Later, several generalizations of this principle were introduced by Kannan (Kannan 1968), Chatterjea (Chatterjea 1972), Reich (Reich 1971), Hardy–Rogers (Hardy and Rogers 1973), and Ćirić (Ćirić 1974). These developments led to the study of broader classes of mappings and different iterative methods.

Unlike contractive mappings, nonexpansive mappings do not always guarantee the existence of fixed points. For this reason, fixed-point theory for nonexpansive mappings has been studied under additional geometric assumptions. Important existence results were obtained by Browder (Browder 1965), Göhde (Göhde 1965), and Kirk (Kirk 1965) in uniformly convex Banach spaces. Later, the theory was extended to nonlinear settings such as geodesic spaces, hyperbolic spaces, and $CAT(0)$ spaces. In this direction, Takahashi (Takahashi 1970), Goebel and Kirk (Goebel and Kirk 1983), and Kohlenbach (Kohlenbach 2005) made significant contributions to the development of fixed-point theory in nonlinear geometric structures. Moreover, Kirk and Panyanak (Kirk and Panyanak 2008) studied the notion of Δ -convergence in $CAT(0)$ spaces and established several convergence results analogous to weak convergence in Banach spaces. Since then, many convergence results for iterative methods in geodesic and $CAT(0)$ spaces have appeared in the literature (Dhompongsa et al. 2008), (Khan et al. 2011), (Leustean et al. 2010), (Shukla et al. 2022).

Suzuki (Suzuki 2008) introduced Condition-(C), which defines a class of mappings lying between nonexpansive and quasi-nonexpansive mappings. After this work, several generalizations of Condition-(C) were proposed. In particular, Karapınar (Karapınar 2012) introduced the (RSC), (RCSC) and (HRSC)-conditions as generalized versions of Suzuki-type mappings and established some basic properties of these mappings. In recent years, several iterative approximation and convergence results for generalized Suzuki-type mappings have been obtained in $CAT(0)$ and geodesic spaces (Temir 2025), (Saleem et al. 2023), (Shukla 2022).

Fixed point iteration processes are frequently used to approximate fixed points of nonlinear mappings. In 2018, Ullah and Arshad (Ullah and

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Arshad 2018) introduced the M -iteration process and showed that this process has a faster convergence behavior than several classical iteration methods. Afterward, various studies investigated faster iterative schemes and compared their numerical performance (see, e.g., (Alagoz 2018), (Temir and Korkut 2021)). These developments indicate the importance of studying modified M -type iteration processes for broader classes of generalized nonexpansive mappings.

Motivated by these developments, in this paper we study fixed-point approximation results for mappings satisfying the $(HRSC)$ -condition in suitable geodesic spaces. Our aim is to investigate the convergence behavior of the considered iterative process and to extend some existing results in the literature. The obtained results contribute to the study of generalized Suzuki-type mappings in nonlinear geometric settings.

2. Preliminaries

Let $\Omega \neq \emptyset \subseteq \tilde{N}$ where $(\tilde{N}, d)$ is a metric space and $Y: \Omega \rightarrow \Omega$ is a self mapping, then a point $p \in \Omega$ is a fixed point of the mapping Y if $Yp = p$. The set $\mathcal{F}(Y)$, i.e., $\mathcal{F}(Y) = \{p \in \Omega: Yp = p\}$ denotes the set of all fixed points of mapping Y .

Definition 2.1.

1. The mapping Y is called nonexpansive mapping if for $x, y \in \Omega$, we have

$$d(Yx, Yy) \leq d(x, y).$$

2. A self-mapping $Y: \Omega \rightarrow \Omega$ is said to be quasi-nonexpansive mapping if the set of fixed point $\mathcal{F}(Y) \neq \emptyset$ and for all $x \in \Omega$ and $p \in \mathcal{F}(Y)$, we have

$$d(Yx, p) \leq d(x, p).$$

3. (Suzuki 2008) Let $\Omega \neq \emptyset \subseteq \tilde{N}$ where $(\tilde{N}, d)$ is a metric space. Then $Y: \Omega \rightarrow \Omega$ satisfies Condition- (C) if we have

$$\frac{1}{2}d(x, Yx) \leq d(x, y) \Rightarrow d(Yx, Yy) \leq d(x, y)$$

for each $x, y \in \Omega$.

4. (Karapinar 2012) Let $\Omega \neq \emptyset \subseteq \tilde{N}$ (metric space). Then $Y: \Omega \rightarrow \Omega$ satisfies Hardy-Rogers-Suzuki- (C) condition ($(HRSC)$ -condition) if

$$\begin{aligned} \frac{1}{2}d(x, Yx) &\leq d(x, y) \text{ implies that} \\ d(Yx, Yy) &\leq \frac{1}{5} \left[d(x, y) + d(Yx, x) + d(y, Yy) \right. \\ &\quad \left. + d(Yx, y) + d(x, Yy) \right] \\ &\text{for } x, y \in \Omega. \end{aligned}$$

Example 2.2. Let $\Omega = [0, 2]$ with the usual metric $d(x, y) = |x - y|$ and define $Y: \Omega \rightarrow \Omega$ by

$$Y(x) = \begin{cases} 0 & \text{if } x \in [0, 2), \\ 1 & \text{if } x = 2. \end{cases}$$

Then $\mathcal{F}(Y) = \{0\}$. We show that Y satisfies the $(HRSC)$ -condition.

Case 1.

Let $x, y \in [0, 2)$. Then $Y(x) = Y(y) = 0$ and hence the required inequality is immediate.

Case 2.

Let $x = 2$ and $y \in [0, 2)$. If

$$\frac{1}{2}d(2, Y(2)) \leq d(2, y),$$

then $\frac{1}{2} \leq 2 - y$, so $y \leq \frac{3}{2}$. Therefore,

$$d(Y(2), Y(y)) = 1$$

and

$$\begin{aligned} \frac{1}{5} \left[d(2, y) + d(Y(2), 2) + d(y, Y(y)) \right. \\ \left. + d(Y(2), y) + d(2, Y(y)) \right] \\ = \frac{1}{5} [5 + |1 - y|] \geq 1. \end{aligned}$$

Thus the $(HRSC)$ -condition holds.

Case 3.

Let $x \in [0, 2)$ and $y = 2$. If

$$\frac{1}{2}d(x, Y(x)) \leq d(x, 2),$$

then $\frac{x}{2} \leq 2 - x$, so $x \leq \frac{4}{3}$. Hence,

$$d(Y(x), Y(2)) = 1$$

and

$$\begin{aligned} \frac{1}{5} \left[d(x, 2) + d(Y(x), x) + d(2, Y(2)) \right. \\ \left. + d(Y(x), 2) + d(x, Y(2)) \right] \\ = \frac{1}{5} [5 + |x - 1|] \geq 1. \end{aligned}$$

Therefore Y satisfies the $(HRSC)$ -condition.

Moreover, Y does not satisfy the Condition- (C) .

Indeed, for $x = 2$ and $y = \frac{3}{2}$, we have

$$\frac{1}{2}d(2, Y(2)) = \frac{1}{2} \leq d\left(2, \frac{3}{2}\right) = \frac{1}{2},$$

but

$$d\left(Y(2), Y\left(\frac{3}{2}\right)\right) = 1 > d\left(2, \frac{3}{2}\right).$$

Hence Y satisfies the $(HRSC)$ -condition but does not satisfy Condition- (C) .

Remark 2.3. The previous example shows that there exist mappings satisfying the $(HRSC)$ -condition which do not satisfy Condition- (C) . Hence, the $(HRSC)$ -condition is a proper generalization of Condition- (C) .

Let $(\tilde{N}, d)$ be a metric space and let $[0, 1] \subset \mathbb{R}$. For two points $x, y \in \tilde{N}$, a mapping $\xi: [0, 1] \rightarrow \tilde{N}$ is called a path joining x and y whenever $\xi(0) = x$ and $\xi(1) = y$. The path ξ is said to be a geodesic if

$$d(\xi(s), \xi(t)) = d(\xi(0), \xi(1))|s - t|, \text{ for every } s, t \in [0, 1].$$

A metric space $(\tilde{N}, d)$ is called a geodesic space if every pair of points in $\tilde{N}$ can be joined by a geodesic path. In general, the geodesic segment joining two points need not be unique.

The following precise formulation of hyperbolic spaces was introduced by Kohlenbach (Kohlenbach 2005).

Definition 2.4. (Kohlenbach 2005) Let $(\tilde{N}, d)$ be a metric space and the let $W: \tilde{N} \times \tilde{N} \times [0, 1] \rightarrow \tilde{N}$ be a mapping. Then the triplet $(\tilde{N}, d, W)$ is called a W -hyperbolic space if the following conditions hold for all $x, y, z, \omega \in \tilde{N}$ and $\mu, \theta \in [0, 1]$:

1. $d(z, W(x, y, \mu)) \leq (1 - \mu)d(z, x) + \mu d(z, y)$;
2. $d(W(x, y, \mu), W(x, y, \theta)) = |\mu - \theta|d(x, y)$;
3. $W(x, y, \mu) = W(y, x, 1 - \mu)$;
4. $d(W(x, z, \mu), W(y, \omega, \mu)) \leq (1 - \mu)d(x, y) + \mu d(z, \omega)$.

For simplicity, we use the notation

$$W(x, y, \mu) = (1 - \mu)x \oplus \mu y$$

for a point in a W -hyperbolic space. For $x, y \in \tilde{N}$, the geodesic segment joining x and y is denoted by

$$[x, y] = \{(1 - \mu)x \oplus \mu y : \mu \in [0, 1]\}.$$

A subset Ω of W -hyperbolic space $(\tilde{N}, d, W)$ is called convex whenever $[x, y] \subset \Omega$ for every $x, y \in \Omega$.

Definition 2.5. (Leustean 2010) A W -hyperbolic space $(\tilde{N}, d, W)$ is called uniformly convex (UCW -hyperbolic space) if for every $\varepsilon \in (0, 2]$ and $t > 0$, there exists $\delta \in (0, 1]$ such that whenever $x, y, z \in \tilde{N}$ satisfy $d(x, z) \leq t$, $d(y, z) \leq t$ and $d(x, y) \geq \varepsilon t$, then

$$d\left(\frac{1}{2}x \oplus \frac{1}{2}y, z\right) \leq (1 - \delta)t.$$

Let $\{x_k\}$ be a bounded sequence in a hyperbolic space $(\tilde{N}, d, W)$ and Ω a nonempty subset of $\tilde{N}$. A functional $r(\cdot, \{x_k\}): \tilde{N} \rightarrow [0, +\infty)$ can be defined as follows:

$$r(y, \{x_k\}) = \limsup_{k \rightarrow +\infty} d(y, x_k).$$

The asymptotic radius of $\{x_k\}$ with respect to Ω is described as

$$r(\Omega, \{x_k\}) = \inf\{r(y, \{x_k\}): y \in \Omega\}.$$

A point x in Ω is called as an asymptotic center of $\{x_k\}$ with respect to Ω if

$$r(x, \{x_k\}) = r(\Omega, \{x_k\}).$$

$A(\Omega, \{x_k\})$ is denoted as the set of all asymptotic centers of $\{x_k\}$ with respect to Ω .

Definition 2.6. (Sosov 2004) Let $\{x_k\}$ be a bounded sequence in a W -hyperbolic space $(\tilde{N}, d, W)$. The sequence $\{x_k\}$ is said to Δ -converge to a point $x \in \tilde{N}$ if x is the unique asymptotic center of every subsequence of $\{x_k\}$.

Let $(\tilde{N}, d)$ be a W -hyperbolic space and let $\Omega \subset \tilde{N}$ be nonempty. A sequence $\{x_k\}$ in $\tilde{N}$ is called Fejér monotone with respect to Ω if

$$d(p, x_{k+1}) \leq d(p, x_k)$$

for every $p \in \Omega$ and for all $k \geq 0$.

Definition 2.7. (Senter and Dotson 1974) Let $Y: \Omega \rightarrow \Omega$ be a mapping with $\mathcal{F}(Y) \neq \emptyset$. We say that Y satisfies Condition (I) if there exists a function $f: [0, +\infty) \rightarrow [0, +\infty)$ such that:

1. $f(r) > 0$ for all $r > 0$ and $f(0) = 0$.
2. $d(x, Y(x)) \geq f(d(x, \mathcal{F}(Y)))$ for every $x \in \Omega$, where

$$d(x, \mathcal{F}(Y)) = \inf\{d(x, y): y \in \mathcal{F}(Y)\}.$$

Proposition 2.8. (Leustean 2010) Let $(\tilde{N}, d, W)$ be a complete UCW -hyperbolic space and let Ω be a nonempty closed convex subset of $\tilde{N}$. Then every bounded sequence in $\tilde{N}$ admits a unique asymptotic center with respect to Ω .

Lemma 2.9. (Karapinar 2012) Let Ω be a nonempty closed and convex subset of W -hyperbolic space $(\tilde{N}, d, W)$. If $Y: \Omega \rightarrow \Omega$ satisfies the (HRSC)-condition, then

$$d(\rho, Yy) \leq 15d(Y\rho, \rho) + d(\rho, y)$$

for all $\rho, y \in \Omega$

Remark 2.10. Substituting $y = p$ with $p \in \mathcal{F}(Y)$ into the (HRSC)-condition and using $Yp = p$ yields

$$d(Yx, p) \leq \frac{1}{5}[2d(x, p) + d(Yx, x) + d(Yx, p)].$$

Rearranging gives

$$\frac{4}{5}d(Yx, p) \leq \frac{1}{5}[2d(x, p) + d(Yx, x)].$$

Applying the triangle inequality $d(Yx, x) \leq d(Yx, p) + d(x, p)$ leads to

$$d(Yx, p) \leq d(x, p)$$

for all $x \in \Omega$ and $p \in \mathcal{F}(Y)$. Hence Y is quasi-nonexpansive. By induction, we therefore obtain

$$d(Y^k x, p) \leq d(Y^{k-1} x, p) \leq \dots \leq d(Yx, p) \leq d(x, p)$$

for every integer $k \geq 1$.

Lemma 2.11. (Leustean 2010) Let $\{x_k\}$ be a bounded sequence in a W -hyperbolic space $(\tilde{N}, d, W)$ and

assume $A(\Omega, \{x_k\}) = \{z\}$. Let $\{r_k\}$ and $\{s_k\}$ be two real sequences such that $r_k \geq 0$ for all $k \in \mathbb{N}$, $\limsup_{k \rightarrow \infty} r_k \leq 1$ and $\limsup_{k \rightarrow \infty} s_k \leq 0$. Suppose that $y \in \Omega$ and there exist $m, k_0 \in \mathbb{N}$ such that

$$d(y, x_{k+m}) \leq r_k d(z, x_k) + s_k$$

for every $k \geq k_0$. Then, $y = z$.

Lemma 2.12. (Ariza-Ruiz et al. 2014) Let $\{x_k\}$ be a sequence in a W -hyperbolic space $(\tilde{N}, d, W)$ and let $\Omega \subset \tilde{N}$ be nonempty. Assume that $\{x_k\}$ is Fejér monotone with respect to Ω . If $A(\Omega, \{x_k\}) = \{x\}$ and $A(\tilde{N}, \{x_{k_j}\}) \subseteq \Omega$ for every subsequence $\{x_{k_j}\}$ of $\{x_k\}$, then the sequence $\{x_k\}$ Δ -converges to $x \in \Omega$.

Lemma 2.13. (Khan et al. 2012) Let $(\tilde{N}, d, W)$ be a complete UCW -hyperbolic space and let $\omega \in \tilde{N}$. Assume that $\{\alpha_k\} \subset (0, 1)$ satisfies $0 < \alpha \leq \alpha_k \leq \beta < 1$ for all $k \in \mathbb{N}$. Let $\{x_k\}$ and $\{y_k\}$ be sequences in $\tilde{N}$ such that, for some $r \geq 0$,

$$\begin{aligned} \limsup_{k \rightarrow \infty} d(x_k, \omega) &\leq r, \limsup_{k \rightarrow \infty} d(y_k, \omega) \\ &\leq r \text{ and } \lim_{k \rightarrow \infty} d(\alpha_k y_k \\ &\oplus (1 - \alpha_k)x_k, \omega) = r. \end{aligned}$$

Then, $\lim_{k \rightarrow \infty} d(y_k, x_k) = 0$.

3. Main Results

In 2018, Ullah and Arshad (Ullah and Arshad 2018) introduced a three step iterative process called the M -iteration process. Let $\mathcal{B}$ be a Banach space and let Ω be a nonempty convex subset of $\mathcal{B}$. For a mapping $Y: \Omega \rightarrow \Omega$, the M -iteration process is defined by

$$\begin{cases} x_1 \in \Omega \\ x_{k+1} = Y y_k \\ y_k = Y z_k \\ z_k = (1 - \alpha_k)x_k + \alpha_k Y x_k, \end{cases} \quad (1)$$

where $\{\alpha_k\}$ is a sequence in $[0, 1]$ for $k \geq 0$.

The above iteration process can be adapted to the setting of geodesic spaces. Let $(\tilde{N}, d, W)$ be a W -hyperbolic space and let Ω be a nonempty convex subset of $\tilde{N}$. For a mapping $Y: \Omega \rightarrow \Omega$, we consider the following modified M -iteration process:

$$\begin{cases} x_1 \in \Omega \\ x_{k+1} = Y^k y_k \\ y_k = Y^k [(1 - \alpha_k)x_k \oplus \alpha_k Y^k x_k], \end{cases} \quad (2)$$

where $\{\alpha_k\}$ is a sequence in $[0, 1]$ for $k \geq 0$.

Lemma 3.1. Let $(\tilde{N}, d, W)$ be a complete UCW -hyperbolic space and let Ω be a nonempty closed convex subset of $\tilde{N}$. Assume that $Y: \Omega \rightarrow \Omega$ satisfies $(HRSC)$ -condition and that $\mathcal{F}(Y) \neq \emptyset$. Let $\{x_k\}$ be the sequence generated by the iteration process (2), where $\alpha_k \in [\alpha, \beta]$ for some $\alpha, \beta \in (0, 1)$. Then, the following results hold:

1. $\lim_{k \rightarrow \infty} d(x_k, p)$ exists for all $p \in \mathcal{F}(Y)$.
2. $\lim_{k \rightarrow \infty} d(x_k, Y(x_k)) = 0$.

Proof.

1. Let $p \in \mathcal{F}(Y)$. By the convexity of the metric combination and applying the induction-bound from the Remark 2.10 (so that $d(Y^k x, p) \leq d(Yx, p) \leq d(x, p)$ for all $k \geq 1$), we obtain

$$\begin{aligned} d(y_k, p) &= d(Y^k [(1 - \alpha_k)x_k \oplus \alpha_k Y^k x_k], p) \\ &\leq d((1 - \alpha_k)x_k \oplus \alpha_k Y^k x_k, p) \\ &\leq (1 - \alpha_k)d(x_k, p) + \alpha_k d(Y^k x_k, p) \\ &\leq (1 - \alpha_k)d(x_k, p) + \alpha_k d(Yx_k, p) \\ &\leq (1 - \alpha_k)d(x_k, p) + \alpha_k d(x_k, p) \\ &= d(x_k, p). \end{aligned} \quad (3)$$

Using the quasi-nonexpansive property and the inequality (3), we have

$$\begin{aligned} d(x_{k+1}, p) &= d(Y^k y_k, p) \\ &\leq d(y_k, p) \\ &\leq d(x_k, p). \end{aligned} \quad (4)$$

Therefore, the sequence $\{d(x_k, p)\}$ is monotone nonincreasing and bounded. Hence, $\lim_{k \rightarrow \infty} d(x_k, p)$ exists for all $p \in \mathcal{F}(Y)$.

2. Assume that,

$$\lim_{k \rightarrow \infty} d(x_k, p) = h > 0.$$

By (i), we get

$$d(y_k, p) \leq d(x_k, p)$$

and taking limsup on both sides, we have

$$\limsup_{k \rightarrow \infty} d(y_k, p) \leq \limsup_{k \rightarrow \infty} d(x_k, p) = h. \quad (5)$$

From inequality (4), we know that

$$d(x_{k+1}, p) \leq d(y_k, p).$$

Taking liminf on both sides, we get

$$h = \liminf_{k \rightarrow \infty} d(x_{k+1}, p) \leq \liminf_{k \rightarrow \infty} d(y_k, p). \quad (6)$$

Using inequalities (5) and (6), we have

$$\lim_{k \rightarrow \infty} d(y_k, p) = h.$$

From the foregoing inequality and the quasi-nonexpansive property in the Remark 2.10, we also have

$$\begin{aligned} h &= \lim_{k \rightarrow \infty} d(y_k, p) \\ &= \lim_{k \rightarrow \infty} d(Y^k [(1 - \alpha_k)x_k \oplus \alpha_k Y^k x_k], p) \\ &\leq \lim_{k \rightarrow \infty} d((1 - \alpha_k)x_k \oplus \alpha_k Y^k x_k, p) \\ &\leq \lim_{k \rightarrow \infty} [(1 - \alpha_k)d(x_k, p) + \alpha_k d(Y^k x_k, p)] \\ &\leq \lim_{k \rightarrow \infty} [(1 - \alpha_k)d(x_k, p) + \alpha_k d(Yx_k, p)] \\ &\leq \lim_{k \rightarrow \infty} [(1 - \alpha_k)d(x_k, p) + \alpha_k d(x_k, p)] \end{aligned} \quad (7)$$

$$= \lim_{k \rightarrow \infty} d(x_k, p) \\ = h.$$

Therefore, from the line (7) in the inequality above,

$$\lim_{k \rightarrow \infty} d((1 - \alpha_k)x_k \oplus \alpha_k Yx_k, p) = h.$$

Using Lemma 2.13, we obtain

$$\lim_{k \rightarrow +\infty} d(x_k, Y(x_k)) = 0.$$

□

We now establish the Δ -convergence result for mappings satisfying the (HRSC)-condition in complete UCW-hyperbolic spaces.

Theorem 3.2. Let $\tilde{N}, \Omega, Y$, and $\{x_k\}$ be as in Lemma 3.1. Then, the sequence $\{x_k\}$ Δ -converges to a point in $\mathcal{F}(Y)$.

Proof. *Proof.* By Lemma 3.1, the sequence $\{d(x_k, z^\dagger)\}$ is nonincreasing for every $z^\dagger \in \mathcal{F}(Y)$. Hence $\{x_k\}$ is Fejér monotone with respect to $\mathcal{F}(Y)$. Since $\mathcal{F}(Y)$ is closed and convex (Karapinar 2012), Proposition 2.8 guarantees that $\{x_k\}$ admits a unique asymptotic center $\omega^\dagger$ with respect to $\mathcal{F}(Y)$. Let $\{x_{k_j}\}$ be an arbitrary subsequence of $\{x_k\}$. Again by Proposition 2.8, the sequence $\{x_{k_j}\}$ has a unique asymptotic center $\rho^\dagger$ with respect to $\mathcal{F}(Y)$. Now,

$$\begin{aligned} d(x_{k_j}, Y\rho^\dagger) &\leq d(Yx_{k_j}, Y\rho^\dagger) + d(Yx_{k_j}, x_{k_j}) \\ &\leq \frac{1}{5} \left[d(x_{k_j}, \rho^\dagger) + d(Yx_{k_j}, x_{k_j}) \right] \\ &\quad + d(\rho^\dagger, Y\rho^\dagger) + d(Yx_{k_j}, \rho^\dagger) \\ &\quad + d(x_{k_j}, Y\rho^\dagger) \\ &\quad + d(Yx_{k_j}, x_{k_j}) \\ &\leq \frac{1}{5} \left[3d(x_{k_j}, \rho^\dagger) + 2d(Yx_{k_j}, x_{k_j}) \right] \\ &\quad + 2d(x_{k_j}, Y\rho^\dagger) \\ &\quad + d(Yx_{k_j}, x_{k_j}) \end{aligned}$$

and thus, we have

$$d(x_{k_j}, Y\rho^\dagger) \leq d(x_{k_j}, \rho^\dagger) + \frac{7}{3} d(Yx_{k_j}, x_{k_j}).$$

Using Lemma 3.1 (ii) together with Lemma 2.11, we obtain $Y(\rho^\dagger) = \rho^\dagger$. Therefore $\rho^\dagger \in \mathcal{F}(Y)$. Applying Lemma 2.12, it follows that the sequence $\{x_k\}$ Δ -converges to a point of $\mathcal{F}(Y)$. □

Next, we prove a strong convergence result in the compact setting.

Theorem 3.3. Let $(\tilde{N}, d, W)$ be a complete UCW-hyperbolic space and $\Omega \neq \emptyset$ is a closed convex and compact subset of $\tilde{N}$. Let $Y: \Omega \rightarrow \Omega$ be a self mapping

satisfying (HRSC)-condition and $\mathcal{F}(Y) \neq \emptyset$. Then the sequence $\{x_k\}$ defined by (2) strongly converges to a fixed point of Y .

Proof. *Proof.* From Lemma 3.1, $\lim_{k \rightarrow +\infty} d(x_k, Y(x_k)) = 0$. Since Ω is compact, $\{x_k\}$ has a subsequence $\{x_{k_j}\}$ such that $\{x_{k_j}\}$ converges strongly to $p \in \Omega$.

Using Lemma 2.9, we have

$$d(x_{k_j}, Yp) \leq 15d(x_{k_j}, Yx_{k_j}) + d(x_{k_j}, p).$$

Taking limit as $j \rightarrow \infty$, we obtain that the sequence $\{x_{k_j}\}$ converges to Yp , so $Yp = p$ i.e., $p \in \mathcal{F}(Y)$. Lemma 3.1 suggests that $\lim_{k \rightarrow +\infty} d(x_k, p)$ exists. Therefore, the sequence $\{x_k\}$ converges to p . □

Finally, we establish a strong convergence theorem under Condition (I).

Theorem 3.4. Let $\tilde{N}, \Omega, Y$, and $\{x_k\}$ be same as in Lemma 3.1. If the mapping Y has Condition (I), then the sequence $\{x_k\}$ strongly converges to a point in $\mathcal{F}(Y)$.

Proof. *Proof.* By Lemma 3.1, the sequences $\{d(x_k, z^\dagger)\}$ are monotone nonincreasing for all $z^\dagger \in \mathcal{F}(Y)$. Thus, the sequence $\{d(x_k, \mathcal{F}(Y))\}$ is monotone nonincreasing. Hence, $\lim_{k \rightarrow +\infty} d(x_k, \mathcal{F}(Y))$ exists. From Lemma 3.1,

$$\lim_{k \rightarrow +\infty} d(x_k, Y(x_k)) = 0 \quad (8)$$

Since Y satisfies Condition (I),

$$d(x_k, Y(x_k)) \geq \mathcal{F}(d(x_k, \mathcal{F}(Y)))$$

From (8), $\lim_{k \rightarrow +\infty} \mathcal{F}(d(x_k, \mathcal{F}(Y))) = 0$ and hence

$$\lim_{k \rightarrow +\infty} d(x_k, \mathcal{F}(Y)) = 0. \quad (9)$$

Now, one can verify that the sequence $\{x_k\}$ is Cauchy. For a given $\varepsilon > 0$, from (9), there exists a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$

$$d(x_k, \mathcal{F}(Y)) < \frac{\varepsilon}{4}$$

and

$$\inf\{d(x_{k_0}, z^\dagger) : z^\dagger \in \mathcal{F}(Y)\} < \frac{\varepsilon}{4},$$

and there exists $z^\dagger \in \mathcal{F}(Y)$ such that

$$d(x_{k_0}, z^\dagger) < \frac{\varepsilon}{2}.$$

Therefore, for all $m, k \geq k_0$,

$$\begin{aligned} d(x_{k+m}, x_k) &\leq d(x_{k+m}, z^\dagger) + d(z^\dagger, x_k) \\ &\leq 2d(x_{k_0}, z^\dagger) < 2 \cdot \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

and thus, the sequence $\{x_k\}$ is Cauchy. By the closedness of the set Ω in complete UCW-hyperbolic

space $\tilde{N}$, the sequence $\{x_k\}$ converges to a point $p \in \Omega$. Moreover, since $\lim_{k \rightarrow +\infty} d(x_k, \mathcal{F}(Y)) = 0$, we obtain $d(p, \mathcal{F}(Y)) = 0$. Since $\mathcal{F}(Y)$ is closed, it follows that $p \in \mathcal{F}(Y)$. Thus, $p = Y(p)$. Therefore, the sequence $\{x_k\}$ strongly converges to a point in $\mathcal{F}(Y)$. $\square$

The following example illustrates the applicability of Theorem 3.3 to mappings satisfying the (HRSC)-condition which are not necessarily mappings satisfying Condition-(C).

Example 3.5. Let $\tilde{N} = \mathbb{N}$ be endowed with the usual metric $d(x, y) = |x - y|$ and let $\Omega = [0, 10]$. Since $\mathbb{N}$ is a $CAT(0)$ space and Ω is a nonempty closed convex compact subset of $\mathbb{N}$, Ω is a complete UCW-hyperbolic space with the usual convex structure

$$W(x, y, \mu) = (1 - \mu)x \oplus \mu y, \text{ for all } x, y \in \Omega, \mu \in [0, 1].$$

Define $Y: \Omega \rightarrow \Omega$ by

$$Y(x) = \begin{cases} 6 & \text{if } x = 10, \\ \frac{x+4}{4} & \text{if } x \neq 10. \end{cases}$$

Clearly, $Y(\Omega) \subseteq \Omega$. Moreover, $\mathcal{F}(Y) = \{\frac{4}{3}\}$. Indeed, if $x \neq 10$, then $Y(x) = x$ gives

$$\frac{x+4}{4} = x,$$

and hence $x = \frac{4}{3}$. Also, $Y(10) = 6 \neq 10$.

We now show that Y satisfies the (HRSC)-condition. Let $x, y \in \Omega$. We consider the following cases:

Case 1.

Let $x, y \in [0, 10)$. Then $Y(x) = \frac{x+4}{4}$, $Y(y) = \frac{y+4}{4}$ and hence

$$|Y(x) - Y(y)| = \frac{1}{4}|x - y|.$$

Moreover,

$$\begin{aligned} |x - Y(y)| + |y - Y(x)| &\geq |(x - Y(y)) + (y - Y(x))| \\ &= \left| x - \frac{y+4}{4} - y + \frac{x+4}{4} \right| \\ &= \frac{5}{4}|x - y|. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \frac{1}{5} \left[\frac{|x - y| + |x - Y(x)|}{|y - Y(y)| + |Y(x) - y|} \right] &\geq \frac{1}{5} \left[\frac{|x - y| + |Y(x) - y|}{|x - Y(y)|} \right] \\ &\geq \frac{1}{5} \left[|x - y| + \frac{5}{4}|x - y| \right] \\ &= \frac{9}{20}|x - y| \\ &\geq \frac{1}{4}|x - y| \\ &= |Y(x) - Y(y)|. \end{aligned}$$

Hence, the (HRSC)-condition holds in this case.

Case 2.

Let $x = 10$ and $y \in [0, 10)$. Assume that $\frac{1}{2}|x - Y(x)| \leq |x - y|$. Since $x = 10$ and $Y(10) = 6$, we get $2 \leq 10 - y$, and hence $y \leq 8$.

For $y \in [0, 8]$, it is easy to see that

$$|3y - 4| + 4|6 - y| \geq 8.$$

Therefore, we obtain

$$\begin{aligned} \frac{1}{5} \left[\frac{|10 - y| + |10 - Y(10)|}{|y - Y(y)| + |Y(10) - y|} \right] &= \frac{1}{5} \left[\frac{10 - y + 4 + \frac{|3y - 4|}{4}}{|6 - y| + \frac{36 - y}{4}} \right] \\ &= \frac{1}{20} \left[\frac{76 - 5y + |3y - 4|}{|6 - y|} \right] \\ &\geq \frac{1}{20} [76 - 5y + 8] \\ &= \frac{84 - 5y}{20} \\ &\geq \frac{20 - y}{4} \\ &= |Y(10) - Y(y)|. \end{aligned}$$

Thus the (HRSC)-condition holds.

Case 3.

Let $x \in [0, 10)$ and $y = 10$. Assume that $\frac{1}{2}|x - Y(x)| \leq |x - 10|$. That is,

$$\frac{1}{2} \left| x - \frac{x+4}{4} \right| = \frac{|3x - 4|}{8} \leq 10 - x.$$

This implies $x \leq \frac{84}{11} < 8$.

Now

$$|Y(x) - Y(10)| = \left| \frac{x+4}{4} - 6 \right| = \frac{20 - x}{4}.$$

Moreover, for $x \in [0, \frac{84}{11}]$, we have

$$|3x - 4| + 4|x - 6| \geq 8.$$

Therefore, we obtain

$$\begin{aligned} \frac{1}{5} \left[\begin{array}{c} |x-10| + |x-Y(x)| \\ + |10-Y(10)| + |Y(x)-10| \\ + |x-Y(10)| \end{array} \right] &= \frac{1}{5} \left[\begin{array}{c} 10-x \\ |3x-4| \\ +4 + \frac{36-x}{4} \\ + |x-6| \end{array} \right] \\ &= \frac{1}{20} \left[\begin{array}{c} 76-5x \\ + |3x-4| \\ + 4|x-6| \end{array} \right] \\ &\geq \frac{1}{20} [76-5x+8] \\ &= \frac{84-5x}{20} \\ &\geq \frac{20-x}{4} \\ &= |Y(x)-Y(10)|. \end{aligned}$$

Thus, the (HRSC)-condition holds in this case.

Therefore, Y satisfies the (HRSC)-condition on Ω .

Moreover, Y does not satisfy the Condition-(C). Indeed, take $x = 7$ and $y = 10$, we have

$$\frac{1}{2} |x - Y(x)| = \frac{1}{2} \left| 7 - \frac{7+4}{2} \right| = \frac{17}{8} < 3 = |x - y|.$$

However,

$$|Y(x) - Y(y)| = \left| \frac{11}{4} - 6 \right| = \frac{13}{4} > 3 = |x - y|.$$

Hence Y satisfies the (HRSC)-condition but does not satisfy Condition-(C).

Since Ω is closed, convex and compact, and $\mathcal{F}(Y) \neq \emptyset$ all assumptions of Theorem 3.3 are satisfied. Therefore, the sequence $\{x_k\}$ generated by the iterative process (2) strongly converges to the fixed point $\frac{4}{3}$.

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