Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 55 (3) (2026), 1087 – 1104
DOI : 10.15672/hujms.1693527

Research Article

Screen generic transversal lightlike submanifolds

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Abstract

In this study, we introduce screen generic transversal lightlike submanifolds of indefinite Kaehler manifolds. We show that this new class is an umbrella of invariant (complex), screen real, SCR, radical ST and ST anti-invariant lightlike submanifolds. We investigate the integrability of various distributions and prove a characterization theorem of such lightlike submanifolds in a complex space form. We obtain new conditions for the induced connection to be a metric connection and we find some necessary and sufficient conditions for minimal screen generic transversal lightlike submanifolds. Moreover, we provide some examples of screen generic transversal lightlike submanifold and minimal screen generic transversal lightlike submanifold of an indefinite Kaehler manifold.

Mathematics Subject Classification (2020). 53C15, 53C40, 53C50

Keywords. indefinite Kaehler manifold, lightlike submanifold, screen generic transversal lightlike submanifold, minimal lightlike submanifold

1. Introduction and preliminaries

In differential geometry, K. L. Duggal-A. Bejancu [5] and K. L. Duggal and B. Şahin [7] initiated the geometry of lightlike submanifolds of semi-Riemannian manifolds. This kind of submanifolds are different than non-degenerate submanifolds. Due to the difference of lightlike submanifolds, the study of lightlike submanifolds becomes more difficult and interesting from the study of non-degenerate submanifolds.

Let $(\bar{M}, \bar{g})$ be a real $(m+n)-$dimensional semi-Riemannian manifold of constant index q, such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an $m-$dimensional submanifold of $(\bar{M}, \bar{g})$, where g is the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M then M is named a lightlike submanifold of $(\bar{M}, \bar{g})$. For a degenerate metric g on M

$$TM^{\perp} = \cup \left\{ u \in T_x\bar{M} : \bar{g}(u, v) = 0, \forall v \in T_xM, x \in M \right\}$$

is a degenerate $n-$dimensional subspace of $T_x\bar{M}$. Hence, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. Thus, there exists a subspace $Rad(T_xM) = T_xM \cap T_xM^{\perp}$ which is known as radical (null) space. If the mapping

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Received: 13.05.2025; Accepted: 21.11.2025

$Rad(TM) : x \in M \longrightarrow Rad(T_xM)$, defines a smooth distribution, named radical distribution on M of rank $r > 0$ then the submanifold M of $(\bar{M}, \bar{g})$ is named an $r-$lightlike submanifold.

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM. This means that

$$TM = S(TM) \perp Rad(TM) \tag{1.1}$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $Rad(TM)$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}_{|_M}$ and $Rad(TM)$ in $S(TM^{\perp})^{\perp}$, respectively. Then, we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}), \tag{1.2}$$

$$T\bar{M}\,|_M = TM \oplus tr(TM) = \{Rad(TM) \oplus ltr(TM)\} \perp S(TM) \perp S(TM^{\perp}). \tag{1.3}$$

By taking into account the definition of lightlike submanifolds, we have the following different types of lightlike submanifolds; CR-lightlike submanifolds [5], contact CR-lightlike submanifolds [10], screen Cauchy-Riemann (SCR)-lightlike submanifolds [8], contact SCR-lightlike submanifolds [10], generalized Cauchy-Riemann (GCR)-lightlike submanifolds [9, 11], screen transversal lightlike submanifolds [23, 24], generic lightlike submanifolds [16], screen generic lightlike submanifolds [3, 12, 21], generalized screen generic lightlike submanifolds [22]. More studies for this kind of submanifolds see also: [2, 13–15, 17–20].

The purpose of the present paper is to introduce generalized screen generic transversal lightlike submanifolds of indefinite Kaehler manifolds. We show that this new class is an umbrella of invariant (complex), screen real, SCR, radical ST and ST anti-invariant lightlike submanifolds.

The paper is organized as follows: In Section 2, we introduce screen generic transversal lightlike submanifolds of indefinite Kaehler manifolds. We show that this new class is an umbrella of invariant (complex), screen real, SCR, radical ST and ST anti-invariant lightlike submanifolds and give an non-trival example for the existence of the new submanifolds. Moreover, we investigate the integrability of various distributions and prove a characterization theorem of such lightlike submanifolds in a complex space form and obtain new conditions for the induced connection to be a metric connection. In section 3, we find some necessary and sufficient conditions for minimal screen generic transversal lightlike submanifolds. Moreover, we provide an example for minimal screen generic transversal lightlike submanifold of an indefinite Kaehler manifold.

Now, we are going to mention some theorems and notions which will be used for next sections.

Theorem 1.1 ([4]). *Let $(M, g, S(TM), S(TM^{\perp}))$ be an $r-$lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Suppose U is a coordinate neighbourhood of M and ξ_j, $j \in \{1, .., r\}$ is a basis of $\Gamma(Rad(TM)_{|_U})$. Then, there exist a complementary vector subbundle $ltr(TM)$ of $Rad(TM)$ in $S(TM^{\perp})^{\perp}_{|_U}$ and a basis $\{N_i\}$, $i \in \{1, .., r\}$ of $\Gamma(ltr(TM)_{|_U})$ such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0 \tag{1.4}$$

for any $i, j \in \{1, .., r\}$.

We say that a submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $(\bar{M}, \bar{g})$ is
Case 1: $r-$lightlike if $r < min\{m, n\}$,
Case 2: Coisotropic if $r = n < m$, $S\left(TM^{\perp}\right) = \{0\}$,
Case 3: Isotropic if $r = m < n$, $S(TM) = \{0\}$,
Case 4: Totally lightlike if $r = m = n$, $S(TM) = \{0\} = S(TM^{\perp})$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $(\bar{M}, \bar{g})$. Then, using (1.3), we have

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{1.5}$$
$$\bar{\nabla}_X \bar{N} = -A_{\bar{N}} X + \nabla^t_X \bar{N}, \tag{1.6}$$

for any $X, Y \in \Gamma(TM)$ and $\bar{N} \in \Gamma(tr\,(TM))$, where $\{\nabla_X Y, A_{\bar{N}} X\}$ and $\left\{h(X,Y), \nabla^t_X \bar{N}\right\}$ belong to $\Gamma(TM)$ and $\Gamma(tr\,(TM))$, respectively. ∇ and ∇^t are linear connections on M and on the vector bundle $tr\,(TM)$, respectively. According to (1.2), considering the projection morphisms L and S of $tr\,(TM)$ on $ltr\,(TM)$ and $S(TM^{\perp})$, respectively, (1.5) and (1.6) become

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \tag{1.7}$$
$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N), \tag{1.8}$$
$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W), \tag{1.9}$$

for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr\,(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, where $h^l(X,Y) = Lh(X,Y)$, $h^s(X,Y) = Sh(X,Y)$, $\nabla_X Y, A_N X, A_W X \in \Gamma(TM)$, $\nabla^s_X W, D^s(X,N) \in \Gamma((STM^{\perp}))$ and $\nabla^l_X N, D^l(X,W) \in \Gamma(ltr\,(TM))$. Hence, using (1.7)-(1.9) and letting into account that $\bar{\nabla}$ is a metric connection we derive

$$g(h^s(X,Y),W) + g(Y, D^l(X,W)) = g(A_W X, Y), \tag{1.10}$$
$$g(D^s(X,N),W) = g(A_W X, N), \tag{1.11}$$
$$g(h^l(X,Y),\xi) + g(Y, h^l(X,\xi)) + g(Y, \nabla_X \xi) = 0. \tag{1.12}$$

Let Q be a projection of TM on $S(TM)$. Thus, using (1.1) we obtain

$$\nabla_X QY = \nabla^*_X QY + h^*(X, QY), \tag{1.13}$$
$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{1.14}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\{\nabla^*_X QY, A^*_\xi X\}$ and $\{h^*(X,QY), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S\,(TM))$ and $\Gamma(Rad\,(TM))$, respectively.

Using the equations given above, we derive

$$g(h^l(X,QY),\xi) = g(A^*_\xi X, QY), \tag{1.15}$$
$$g(h^*(X,QY),N) = g(A_N X, QY), \tag{1.16}$$
$$g(h^l(X,\xi),\xi) = 0, \quad A^*_\xi \xi = 0. \tag{1.17}$$

Generally, ∇ on M is not metric connection. Since $\bar{\nabla}$ is a metric connection, from (1.7), we obtain

$$(\nabla_X g)(Y,Z) = \bar{g}(h^l(X,Y),Z) + \bar{g}(h^l(X,Z),Y).$$

However, it is important to note that $\nabla^\star$ is a metric connection on $S(TM)$. We denote curvature tensor of a lightlike submanifold by R, then the Gauss equation for lightlike submanifolds is given by

$$\begin{aligned}
\bar{R}(X,Y)Z &= R(X,Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)} + A_{h^s(X,Z)}Y \\
&- A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y,Z) - (\nabla_Y h^l)(X,Z) \\
&+ D^l(X, h^s(Y,Z)) - D^l(Y, h^s(X,Z)) \\
&+ (\nabla_X h^s)(Y,Z) - (\nabla_Y h^s)(X,Z) \\
&+ D^s(X, h^l(Y,Z)) - D^s(Y, h^l(X,Z))
\end{aligned} \tag{1.18}$$

for any $X, Y, Z \in \Gamma(TM)$.

Theorem 1.2. *Let M be an $r-$lightlike submanifold of a semi-Riemannian manifold $\bar{M}$. Then ∇ is a metric connection iff $Rad(TM)$ is a parallel distribution with respect to ∇ [4].*

Definition 1.3. A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be totally umbilical in $\bar{M}$ if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, called the transversal curvature vector field of M, such that

$$h(X, Y) = Hg(X, Y) \tag{1.19}$$

for any $X, Y \in \Gamma(TM)$. In case $H = 0$, M is called totally geodesic [6].

Using (1.7) and (1.19) it is easy to see that M is totally umbilical iff on each coordinate neighborhood U there exists smooth vector fields $H^l \in \Gamma(ltr\,(TM))$ and $H^s \in \Gamma((STM^{\perp}))$ such that

$$h^l(X, Y) = g(X, Y)H^l,\ h^s(X, Y) = g(X, Y)H^s \text{ and } D^l(X, W) = 0 \tag{1.20}$$

for any $X, Y \in \Gamma(TM)$ and $W \in \Gamma((STM^{\perp}))$.

Now, we recall some basic definitions of indefinite Kähler manifolds. Let $(\bar{M}, \bar{g})$ be a $2k$-dimensional semi-Riemannian manifold and the constant index of $\bar{g}$ be q, $0 < q < 2k$. An almost complex structure J on $\bar{M}$ is a tensor field of type (1,1) and an endomorphism of $T_p\bar{M}$, such that $J^2 = -I$, $\forall p \in \bar{M}$, where I denotes the identity transformation of $T_p\bar{M}$. Then, $(\bar{M}, J, \bar{g})$ is called an almost complex manifold. If semi-Riemannian metric $\bar{g}$ of an almost complex manifold $\bar{M}$ satisfies the following conditions, then $(\bar{M}, J, \bar{g})$ is an indefinite almost Hermitian manifold

$$\bar{g}(X, Y) = \bar{g}(JX, JY)\,, \quad \forall X, Y \in \Gamma(T\bar{M}). \tag{1.21}$$

Let $\bar{\nabla}$ be the Levi-Civita connection on an indefinite almost Hermitian manifold $(\bar{M}, J, \bar{g})$ and J be parallel with respect to $\bar{\nabla}$, i.e.,

$$(\bar{\nabla}_X J)Y = 0\,, \quad \forall X, Y \in \Gamma(T\bar{M}). \tag{1.22}$$

Then $\bar{M}$ is called an indefinite Kähler manifold.
If holomorphic sectional curvature of an indefinite Kähler manifold is constant c, then it is called an indefinite complex space form and denoted by $\bar{M}(c)$. The curvature tensor field of $\bar{M}(c)$ is given by

$$\begin{aligned} \bar{R}(X, Y)Z &= \frac{c}{4}\{\bar{g}(Y, Z)X - \bar{g}(X, Z)Y + \bar{g}(JY, Z)JX - \bar{g}(JX, Z)JY \\ &+ 2\bar{g}(X, JY)JZ\}, \end{aligned} \tag{1.23}$$

for any $X, Y \in \Gamma(T\bar{M})$ [1].

2. Screen generic transversal lightlike submanifolds

Definition 2.1. Let M be a real r-lightlike submanifold of an indefinite Kaehler manifold manifold $(\bar{M}, \bar{g})$. Then we say that M is a screen generic transversal lightlike submanifold if the condition (A) and (B) are holded:

(A) There exist two subbundles D_1 and D_2 of $Rad(TM)$ such that

$$Rad(TM) = D_1 \oplus D_2,\ \ J(D_1) = D_1,\ J(D_2) \subset S(TM^{\perp}). \tag{2.1}$$

(B) There exists a subbundle D_0 of $S\,(TM)$ such that

$$D_0 = J(S(TM)) \cap S(TM), \tag{2.2}$$

where D_0 is a non-degenerate distribution on M.

From definition of a screen generic transversal lightlike submanifold, we obtain that

$$S(TM) = D_0 \oplus D' \tag{2.3}$$

where D' is a complementary non-degenerate distribution to D_0 in $S(TM)$ such that

$$J(D^{'}) \nsubseteq S(TM) \;\; and \;\; J(D^{'}) \nsubseteq S(TM^{\perp}).$$

Furthermore, TM of M is decomposed as

$$TM = D \oplus D^{\mu} \tag{2.4}$$

where

$$D = D_0 \oplus D_1, \tag{2.5}$$

$$D^{\mu} = D' \oplus D_2. \tag{2.6}$$

Thus D is invariant and we see that neither D^{μ} is invariant nor D^{μ} is anti-invariant. Besides, we have

$$ltr(TM) = L_1 \oplus L_2, J(L_1) = L_1, J(L_2) \subset S(TM^{\perp}). \tag{2.7}$$

Now, we denote the projections from $\Gamma(TM)$ to $\Gamma(D')$ and $\Gamma(D_2)$ by P_1 and P_2; respectively. Thus, we write

$$X = QX + P_1X + P_2X \tag{2.8}$$

$$= QX + PX \tag{2.9}$$

where $QX \in \Gamma(D)$ and $PX \in \Gamma(D^{\mu})$. From the last equation, we obtain

$$JX = \Phi X + \omega P_1X + \omega P_2X \tag{2.10}$$

$$= \Phi X + \omega X \tag{2.11}$$

for any $X \in \Gamma(TM)$; where ΦX and ωX are tangential and transversal parts of JX, respectively. Besides, it is clear that $J(D^{'}) \neq D^{'}$.
On the other hand, for a vector field $Y \in \Gamma(D^{'})$, we have

$$JY = \Phi Y + \omega Y \tag{2.12}$$

such that, $\Phi Y \in \Gamma(D^{'})$ and $\omega Y \in \Gamma(S(TM^{\perp}))$.
Similarly, for $W \in \Gamma(tr(TM))$, we get following decomposition

$$JW = BW + CW \tag{2.13}$$

where BW is tangential part and CW is transversal part of JW, respectively.

If $D_1 \neq \{0\}$, $D_2 \neq \{0\}$, $D_0 \neq \{0\}$ and $D' \neq \{0\}$, then M is called a proper screen generic transversal lightlike submanifold of an indefinite Kaehler manifold $(\bar{M}, \bar{g})$. For proper screen generic transversal lightlike submanifold we note that the following features:

1. The condition (A) implies that $dim(Rad(TM) \geq 3$.
2. The condition (B) implies $dim(D_0) = 2s \geq 2$, $dim(D') = 2p \geq 2$ and $dim(D_2) = dim(L_2)$. Thus $dim(M) \geq 7$ and $dim(\bar{M}) \geq 14$.
3. Any proper 7-dimensional screen generic transversal lightlike submanifold must be 3-lightlike.
4. (A) imply that index$(\bar{M}) \geq 3$.

Proposition 2.2. *A screen generic transversal lightlike submanifold M of an indefinite Kaehler manifold $(\bar{M}, \bar{g})$ is screen generic lightlike submanifold iff $D_2 = \{0\}$.*

We easily observe that

(a) If $D_2 = D' = \{0\}$, then M is an invariant lightlike submanifold.

(b) If D' is anti-invariant and $D_2 = D_0 = \{0\}$, then M is screen real lightlike submanifold.

(c) If D' is anti-invariant and $D_2 = \{0\}$, then M is SCR-lightlike submanifold.

(d) If $D_1 = D' = \{0\}$, then M is radical ST lightlike submanifold.

(e) If $D_1 = D = \{0\}$ and D' anti-invariant, then M is ST anti-invariant lightlike submanifold.

(f) If $D_2 = \{0\}$, then M is screen generic lightlike submanifold.

Proposition 2.3. *There exist no coisotropic, isotropic or totally lightlike proper screen generic transversal lightlike submanifolds M of an indefinite Kaehler manifold. Any coisotropic or totally lightlike screen generic transversal lightlike submanifold M is an invariant submanifold. Furthermore, a isotropic screen generic transversal lightlike submanifold is ST lightlike submanifold iff $D_1 = \{0\}$.*

Proof. Assume that M is a proper screen generic transversal lightlike submanifold. From definition of proper screen generic transversal lightlike submanifold, we know that $D_1 \neq \{0\}$, $D_2 \neq \{0\}$, $D_0 \neq \{0\}$ and $D' \neq \{0\}$, that is both $S(TM)$ and $S(TM^{\perp})$ are non-zero. Thus, M can not be a coisotropic, isotropic or totally lightlike submanifold.

On the other hand, if M is coisotropic, then $S(TM^{\perp}) = 0$ which implies that $D_2 = 0$ and $\omega(D^{'}) = 0$,that is, $TM = Rad(TM) = D_1$, which is invariant respect to J.
If M is totally lightlike, then $S(TM) = 0$ and $S(TM^{\perp}) = 0$. Hence, $TM = Rad(TM) = D_1$, which implies M is invariant.
Finally, suppose that M is isotropic. Then $S(TM) = 0$ implies $TM = RadTM$. Thus, if M is ST lightlike submanifold, then $D_1 = \{0\}$. Conversely, if $D_1 = \{0\}$, then $TM = D_2$ i.e. M is ST lightlike submanifold. Thus the proof is completed. □

Example 2.4. Let $(\bar{M} = \mathbb{R}_4^{16}, \bar{g})$ be a semi-Euclidean space, where $\bar{g}$ is of signature $(-,-,-,-,+,+,+,+,+,+,+,+,+,+,+,+)$ with respect to canonical basis $(\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial x_8, \partial x_9, \partial x_{10}, \partial x_{11}, \partial x_{12}, \partial x_{13}, \partial x_{14}, \partial x_{15}, \partial x_{16})$. Consider the complex structure J defined by

$$J(x_1, x_2, ..., x_{15}, x_{16}) = (x_2, -x_1, x_4, -x_3, x_6, -x_5, x_8, -x_7, x_{10}, -x_9, x_{12}, -x_{11}, x_{14}, -x_{13}, x_{16}, -x_{15}).$$

Suppose that M is a submanifold of $\mathbb{R}_4^{16}$ defined by

$$\begin{aligned} x_1 &= u_1 \sin\alpha - u_2 \cos\alpha, x_2 = u_1 \cos\alpha + u_2 \sin\alpha, x_3 = u_3, \\ x_4 &= 0, x_5 = u_3, x_6 = 0, x_7 = -u_2, x_8 = u_1, x_9 = \sin u_4 \sinh u_5, \\ x_{10} &= \cos u_4 \cosh u_5, x_{11} = u_6 - u_7, x_{12} = u_7, x_{13} = u_6 \cos\alpha, \\ x_{14} &= 0, x_{15} = u_6 \sin\alpha, x_{16} = u_7 \sin\alpha. \end{aligned}$$

A local frame of TM is given by

$$\begin{aligned} Z_1 &= \sin\alpha\partial x_1 + \cos\alpha\partial x_2 + \partial x_8, Z_2 = -\cos\alpha\partial x_1 + \sin\alpha\partial x_2 - \partial x_7, \\ Z_3 &= \partial x_3 + \partial x_5, Z_4 = \cos u_4 \sinh u_5 \partial x_9 - \sin u_4 \cosh u_5 \partial x_{10}, \\ Z_5 &= \sin u_4 \cosh u_5 \partial x_9 + \cos u_4 \sinh u_5 \partial x_{10}, \\ Z_6 &= \partial x_{11} + \cosh\alpha\partial x_{13} + \sin\alpha\partial x_{15}, Z_7 = \partial x_{12} + \sinh\alpha\partial x_{16}. \end{aligned}$$

Thus M is a 3−lightlike submanifold of $\mathbb{R}_4^{16}$ with $Rad(TM) = Span\{Z_1, Z_2, Z_3\}$, $D_1 = Span\{Z_1, Z_2\}$, $D_2 = Span\{Z_3\}$. It is easy to see $JZ_1 = Z_2 \in \Gamma(D_1)$, hence $D_1 = Span\{Z_1, Z_2\}$ and $D_2 = Span\{Z_3\}$. On the other hand, since $JZ_4 = Z_5 \in \Gamma(S(TM))$, we derive $D_0 = Span\{Z_4, Z_5\}$. By direct calculations, we derive the lightlike transversal

bundle spanned by

$$N_1 = \frac{1}{2}(-\sin\alpha\partial x_1 - \cos\alpha\partial x_2 + \partial x_8), N_2 = \frac{1}{2}(\cos\alpha\partial x_1 - \sin\alpha\partial x_2 - \partial x_7),$$
$$N_3 = \frac{1}{2}(-\partial x_3 + \partial x_5).$$

Then we see that $L_1 = Span\{N_1, N_2\}$, $L_2 = Span\{N_3\}$ and the screen transversal bundle spanned by

$$W_1 = JZ_3 = \partial x_4 + \partial x_6, W_2 = JN_3 = \frac{1}{2}(-\partial x_4 + \partial x_6),$$
$$W_3 = \cosh\alpha\partial x_{14}, W_4 = -\partial x_{11} + \cosh\alpha\partial x_{13} - \sinh\alpha\partial x_{15},$$
$$W_5 = -\sinh\alpha\partial x_{12} + \partial x_{16}, W_6 = \sinh\alpha\partial x_{11} - \partial x_{15}.$$

Since $JW_3 \neq W_4$, then it is easy to see that $\mu = Span\{W_5, W_6\}$ and

$$JZ_6 = \partial x_{12} + \cosh\alpha\partial x_{14} + \sinh\alpha\partial x_{16} = Z_7 + W_3,$$
$$JZ_7 = -\partial x_{11} - \sinh\alpha\partial x_{15} = \frac{-Z_6 + W_4}{2}.$$

Then $D' = Span\{Z_6, Z_7\}$ and M is a proper screen generic transversal lightlike submanifold of $\mathbb{R}^{16}_4$.

Theorem 2.5. *Let M be a screen generic transversal lightlike submanifold of an indefinite Kaehler manifold $(\bar{M}, \bar{g})$. Then ∇ is a metric connection iff the following conditions are satisfied:*

$$A^*_{JY}X - \nabla_X JY \in \Gamma(J(D_1)), Bh(X, JY) = 0, Y \in \Gamma(D_1),$$
$$A_{JY}X \in \Gamma(D_1), B(D^l(X, JY) + \nabla^s_X JY) = 0, Y \in \Gamma(D_2).$$

Proof. Suppose that ∇ is a metric connection. Using (1.22) we obtain

$$\tilde{\nabla}_X Y = -J\tilde{\nabla}_X JY \tag{2.14}$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(Rad(TM))$. Thus using (1.5) in (2.14) we derive

$$\nabla_X Y + h(X, Y) = -J(\nabla_X JY + h(X, JY)), \tag{2.15}$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(Rad(TM))$. From (1.14), (2.11) and (2.13) in (2.15) and taking the tangential parts from the resulting equation give

$$\nabla_X Y = \Phi A^*_{JY}X - \Phi\nabla^{*t}_X JY - Bh(X, JY), \tag{2.16}$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D_1)$. Similarly, letting $Y \in \Gamma(D_2)$ in (2.14) and using (1.9), (2.11) and (2.13), we derive

$$\nabla_X Y = \Phi A_{JY}X - BD^l(X, JY) - B\nabla^s_X JY. \tag{2.17}$$

Therefore considering Theorem 1.2, the proof comes from (2.16) and (2.17). □

Theorem 2.6. *Let M be a screen generic transversal lightlike submanifold of an indefinite Kaehler manifold $(\bar{M}, \bar{g})$. Then D_0 is integrable iff the following conditions are satisfied:*

$$g(\nabla^*_X JY - \nabla^*_Y JX, JZ) = g(B(h^s(X, JY) - h^s(Y, JX)), Z), \tag{2.18}$$
$$g(h^*(X, JY) - h^*(Y, JX), JN) = g(B(h^s(X, JY) - h^s(Y, JX)), N), \tag{2.19}$$

for any $X, Y \in \Gamma(D_0)$, $Z \in \Gamma(D')$ and $N \in \Gamma(ltr(TM))$.

Proof. From definition of screen generic transversal lightlike submanifold, D_0 is integrable iff for any $X, Y \in \Gamma(D_0)$, $[X, Y] \in \Gamma(D_0)$, that is,

$$g([X, Y], Z) = \bar{g}([X, Y], N) = 0$$

for any $Z \in \Gamma(D^{'})$ and $N \in \Gamma(ltr(TM))$. Using (1.7), (1.13), (1.21), (1.22), (2.11) and (2.13), we obtain

$$g([X,Y],Z) = \bar{g}(\nabla^*_X JY - \nabla^*_Y JX, JZ) - g(B(h^s(X,JY) - h^s(Y,JX)),Z), \tag{2.20}$$

$$\bar{g}([X,Y],N) = g(h^*(X,JY) - h^*(Y,JX),JN) - g(B(h^s(X,JY) - h^s(Y,JX)),N), \tag{2.21}$$

which prove (2.18) and (2.19). This completes the proof. □

Theorem 2.7. *Let M be a screen generic transversal lightlike submanifold of an indefinite Kaehler manifold $(\bar{M},\bar{g})$. Then D is integrable if and only if the following conditions are satisfied:*

$$g(\nabla^*_X JY - \nabla^*_Y JX, \Phi Z) = g(B(h^s(X,JY) - \ h^s(Y,JX)),Z), \tag{2.22}$$

$$h^s(X,JY) - h^s(Y,JX) \in \Gamma(J(L_2) \perp \omega(D') \perp \mu) \tag{2.23}$$

for any $X,Y \in \Gamma(D)$ and $Z \in \Gamma(D^{'})$.

Proof. Let M be a screen generic transversal lightlike submanifold of an indefinite Kaehler manifold $(\bar{M},\bar{g})$. Then D is integrable iff for any $X,Y \in \Gamma(D)$, $[X,Y] \in \Gamma(D)$, that is,

$$g([X,Y],Z) = \bar{g}([X,Y],N_2) = 0$$

for any $Z \in \Gamma(D^{'})$ and $N_2 \in \Gamma(L_2)$. From (1.7), (1.13), (1.21), (1.22), (2.11) and (2.13), we get

$$g([X,Y],Z) = \bar{g}(\nabla^*_X JY - \nabla^*_Y JX, \Phi Z) - g(B(h^s(X,JY) - h^s(Y,JX)),Z), \tag{2.24}$$

$$\bar{g}([X,Y],N_2) = \bar{g}(h^s(X,JY) - h^s(Y,JX), JN_2), \tag{2.25}$$

from the last two equations we get (2.22) and (2.23), which completes the proof. □

Theorem 2.8. *Let M be a screen generic transversal lightlike submanifold of an indefinite Kaehler manifold $(\bar{M},\bar{g})$. Then, $D^{'}$ is integrable iff the following conditions are satisfied:*

$$\nabla_Z \Phi W - \nabla_W \Phi Z - A_{\omega W} Z + A_{\omega Z} W \in \Gamma(D^{'} \perp Rad(TM)), \tag{2.26}$$

$$\bar{g}(A_{wW}Z - A_{wZ}W, JN_1) = \bar{g}(\nabla_Z \Phi W - \nabla_W \Phi Z, JN_1), \tag{2.27}$$

$$\bar{g}(h^s(Z,\Phi W) - h^s(W,\Phi Z), JN_2) = \bar{g}(\nabla^s_W \Phi Z - \nabla^s_Z \Phi W, JN_2) \tag{2.28}$$

for any $Z,W \in \Gamma(D^{'})$, $N_1 \in \Gamma(L_1)$ and $N_2 \in \Gamma(L_2)$.

Proof. From definition of screen generic transversal lightlike submanifold, $D^{'}$ is integrable iff, for any $Z,W \in \Gamma(D^{'})$, $X \in \Gamma(D_0)$, $N_1 \in \Gamma(L_1)$ and $N_2 \in \Gamma(L_2)$,

$$g([Z,W],X) = \bar{g}([Z,W],N_1) = \bar{g}([Z,W],N_2) = 0.$$

Using (1.7), (1.9), (1.21), (1.22) and (2.11), we derive

$$g([Z,W],X) = g(\nabla_Z \Phi W - \nabla_W \Phi Z + A_{\omega Z} W - A_{\omega W} Z, JX), \tag{2.29}$$

$$\bar{g}([Z,W],N_1) = g(A_{wZ}W - A_{wW}Z, JN_1) + \bar{g}(h^l(Z,\Phi W) - h^l(W,\Phi Z), JN_1), \tag{2.30}$$

$$\bar{g}([Z,W],N_2) = \bar{g}(h^s(Z,\Phi W) - h^s(W,\Phi Z), JN_2) + \bar{g}(\nabla^s_Z \Phi W - \nabla^s_W \Phi Z, JN_2). \tag{2.31}$$

Thus the proof follows from equations (2.29)-(2.31), which completes the proof. □

Theorem 2.9. *Let M be a screen generic transversal lightlike submanifold of an indefinite Kaehler manifold $(\bar{M},\bar{g})$. Then, D^{μ} is integrable iff the following conditions are satisfied:*

$$g(\nabla_Z \Phi W - \nabla_W \Phi Z, JX) = g(A_{\omega W} Z - A_{\omega Z} W, JX), \tag{2.32}$$

$$\bar{g}(A_{wW}Z - A_{wZ}W, JN) = \bar{g}(h^*(Z,\Phi W) - h^*(W,\Phi Z), JN) \tag{2.33}$$

for any $Z,W \in \Gamma(D^{\mu})$ and $N \in \Gamma(L_1)$.

Proof. From definition of screen generic transversal lightlike submanifold, D^{μ} is integrable iff, for any $Z, W \in \Gamma(D^{\mu})$, $X \in \Gamma(D_0)$ and $N \in \Gamma(L_1)$,

$$g([Z,W],X) = \bar{g}([Z,W],N) = 0.$$

From (1.7), (1.9), (1.13), (1.21), (1.22) and (2.11), we obtain

$$g([Z,W],X) = g(\nabla_Z \Phi W - \nabla_W \Phi Z + A_{wZ}W - A_{wW}Z, JX), \tag{2.34}$$

$$\bar{g}([Z,W],N) = \bar{g}(h^*(Z,\Phi W) - h^*(W,\Phi Z), JN) - \bar{g}(A_{wW}Z - A_{wZ}W, JN) \tag{2.35}$$

which satisfy (2.32) and (2.33). This completes the proof. □

Theorem 2.10. *Let M be a screen generic transversal lightlike submanifold of an indefinite Kaehler manifold $(\bar{M}, \bar{g})$. Then, D_0 is parallel iff the following conditions are satisfied:*

$$g(\nabla^*_X JY, \Phi Z) = g(Bh^s(X, JY), Z), \tag{2.36}$$

$$\bar{g}(h^*(X, JY), JN) = \bar{g}(Bh^s(X, JY), N) \tag{2.37}$$

for any $X, Y \in \Gamma(D_0)$, $Z \in \Gamma(D')$ and $N \in \Gamma(ltr(TM))$.

Proof. Let M be a screen generic transversal lightlike submanifold of an indefinite Kaehler manifold $(\bar{M}, \bar{g})$. Then, D_0 is parallel iff

$$g(\nabla_X Y, Z) = \bar{g}(\nabla_X Y, N) = 0$$

for any $X, Y \in \Gamma(D_0)$, $Z \in \Gamma(D')$ and $N \in \Gamma(ltr(TM))$. Considering (1.7), (1.13), (1.21), (1.22), (2.11) and (2.13) we get

$$g(\nabla_X Y, Z) = g(\nabla^*_X JY, \Phi Z) - g(Bh^s(X, JY), Z), \tag{2.38}$$

$$\bar{g}(\nabla_X Y, N) = \bar{g}(h^*(X, JY), JN) - \bar{g}(Bh^s(X, JY), N). \tag{2.39}$$

Thus the proof follows from equations (2.38) and (2.39), which completes the proof. □

Theorem 2.11. *Let M be a screen generic transversal lightlike submanifold of an indefinite Kaehler manifold $(\bar{M}, \bar{g})$. Then, D is parallel iff the following conditions are satisfied:*

$$g(\nabla_X JY, \Phi Z) = g(Bh^s(X, JY), Z), \tag{2.40}$$

$$h^s(X, JY) \in \Gamma(\omega(D') \perp J(L_2) \perp \mu) \tag{2.41}$$

for any $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D')$.

Proof. From definition of screen generic transversal lightlike submanifold, D is parallel iff $\nabla_X Y \in \Gamma(D)$ for any $X, Y \in \Gamma(D)$. Using (1.5), (1.22) we get

$$\nabla_X Y = -\Phi \nabla_X \Phi Y - Bh^s(X, \Phi Y). \tag{2.42}$$

Using (1.7), (1.21), (1.22) and (2.11) in (2.42), we derive

$$g(\nabla_X Y, Z) = g(\nabla_X JY, \Phi Z) - g(Bh^s(X, JY), Z), \tag{2.43}$$

$$\bar{g}(\nabla_X Y, N_2) = \bar{g}(h^s(X, JY), JN_2), \tag{2.44}$$

for any $X, Y \in \Gamma(D)$, $Z \in \Gamma(D')$ and $N_2 \in \Gamma(L_2)$, which satisfy (2.40) and (2.41). This completes the proof. □

Theorem 2.12. *Let M be a screen generic transversal lightlike submanifold of an indefinite Kaehler manifold $(\bar{M}, \bar{g})$. Then, D' is parallel iff the following conditions are satisfied:*

$$g(\nabla_Z \Phi W, JX) = g(A_{\omega W}Z, JX), \tag{2.45}$$

$$\bar{g}(A_{\omega W}Z, JN_1) = \bar{g}(\nabla_Z \Phi W, JN_1), \tag{2.46}$$

$$\bar{g}(h^s(Z, \phi W), JN_2) = -\bar{g}(\nabla^s_Z \omega W, JN_2) \tag{2.47}$$

for any $Z, W \in \Gamma(D')$, $X \in \Gamma(D_0)$, $N_1 \in \Gamma(L_1)$ and $N_2 \in \Gamma(L_2)$.

Proof. Suppose that M is a screen generic transversal lightlike submanifold of an indefinite Kaehler manifold $(\bar{M}, \bar{g})$. Then D' is parallel iff

$$g(\nabla_Z W, X) = \bar{g}(\nabla_Z W, N_1) = \bar{g}(\nabla_Z W, N_2) = 0$$

for any $Z, W \in \Gamma(D')$, $X \in \Gamma(D_0)$, $N_1 \in \Gamma(L_1)$ and $N_2 \in \Gamma(L_2)$. Considering (1.7), (1.9), (1.21), (1.22) and (2.11) we derive

$$g(\nabla_Z W, X) = g(\nabla_Z \Phi W - A_{\omega W} Z, JX), \tag{2.48}$$

$$\bar{g}(\nabla_Z W, N_1) = \bar{g}(\nabla_Z \Phi W, JN_1) - \bar{g}(A_{\omega W} Z, JN_1), \tag{2.49}$$

$$\bar{g}(\nabla_Z W, N_2) = \bar{g}(h^s(Z, \Phi W), JN_2) + \bar{g}(\nabla^s_Z \omega W, JN_2). \tag{2.50}$$

Thus the proof follows from equations (2.48)-(2.50), which completes the proof. □

Similar to proof of Theorem 2.10, Theorem 2.11 and Theorem 2.12, we obtain Theorem 2.13.

Theorem 2.13. *Let M be a screen generic transversal lightlike submanifold of an indefinite Kaehler manifold $(\bar{M}, \bar{g})$. Then, D^μ is parallel iff the following conditions are satisfied:*

$$g(\nabla_Z \Phi W, JX) = g(A_{wW} Z, JX), \tag{2.51}$$

$$\bar{g}(A_{wW} Z, JN) = \bar{g}(h^*(Z, \Phi W), JN) \tag{2.52}$$

for any $Z, W \in \Gamma(D^\mu)$, $X \in \Gamma(D_0)$ and $N \in \Gamma(L_1)$.

Theorem 2.14. *Let M be a lightlike submanifold of an indefinite Kaehler manifold $\bar{M}(c)$ with $c \neq 0$. Then, M is a screen generic transversal lightlike submanifold of $\bar{M}(c)$ iff:*

(i) The maximal invariant subspaces of T_pM, $p \in M$, define a distribution

$$D = D_0 \oplus D_1,$$

where $Rad(TM) = D_1 \oplus D_2$ and D_0 is a non-degenerate invariant distribution.

(ii) There exists a lightlike transversal vector bundle $ltr(TM)$ such that

$$\bar{g}(\bar{R}(X, Y)N_1, N_2) = 0,$$

for any $X, Y \in \Gamma(D_0)$, N_1, $N_2 \in \Gamma(ltr(TM))$.

(iii) There exists a non-degenerate distribution D' on M such that

$$\bar{g}(\bar{R}(X, Y)Z, V) \neq 0\,,$$

for any $X, Y \in \Gamma(D)$, $Z, V \in \Gamma(D')$.

(iv) There exists a non-degenerate distribution μ on $S(TM^\perp)$ such that

$$\bar{g}(\bar{R}(X, Y)Z, W) \neq 0,$$

where for any $X, Y \in \Gamma(D)$, $Z \in \Gamma(D')$, $W \in \Gamma(TM^\perp)$, but $W \notin \Gamma(\mu)$.

Proof. Let M be a screen generic transversal lightlike submanifold of $\bar{M}(c)$, $c \neq 0$. From (i), $D = D_0 \oplus D_1$ is maximal invariant subspaces. Next from (1.23), we derive

$$\begin{aligned} \bar{g}(\bar{R}(X, Y)N_1, N_2) &= \frac{c}{4}\begin{bmatrix} \bar{g}(Y, N_1)\bar{g}(X, N_2) - \bar{g}(X, N_1)\bar{g}(Y, N_2) \\ +\bar{g}(JY, N_1)\bar{g}(JX, N_2) - \bar{g}(JX, N_1)\bar{g}(JY, N_2) \\ +2\bar{g}(X, JY)\bar{g}(JN_2, N_2) \end{bmatrix} \\ &= \frac{c}{2}\bar{g}(X, JY)\bar{g}(JN_1, N_2) = 0 \end{aligned}$$

for any $X, Y \in \Gamma(D_0)$, N_1, $N_2 \in \Gamma(ltr(TM))$. Since $g(JX, Y) \neq 0$ and $\bar{g}(JN_1, N_2) = 0$, we get $\bar{g}(\bar{R}(X, Y)N_1, N_2) = 0$. Thus (ii) holds. If we use (1.23), we derive

$$\begin{aligned}\bar{g}(\bar{R}(X, Y)Z, V) &= \frac{c}{4}\left[\begin{array}{c}\bar{g}(Y, Z)\bar{g}(X, V) - \bar{g}(X, Z)\bar{g}(Y, V)\\ +\bar{g}(JY, Z)\bar{g}(JX, V) - \bar{g}(JX, Z)\bar{g}(JY, V)\\ +2\bar{g}(X, JY)\bar{g}(JZ, V)\end{array}\right]\\ &= \frac{c}{2}g(X, JY)\bar{g}(JZ, V) \neq 0\end{aligned}$$

for any $X, Y \in \Gamma(D)$ and $Z, V \in \Gamma(D^{'})$, then (iii) holds.
Similarly, for any $X, Y \in \Gamma(D)$, $Z \in \Gamma(D^{'})$, $W \in \Gamma(TM^{\perp})$, we have

$$\begin{aligned}\bar{g}(\bar{R}(X, Y)Z, W) &= \frac{c}{4}\left[\begin{array}{c}\bar{g}(Y, Z)\bar{g}(X, W) - \bar{g}(X, Z)\bar{g}(Y, W)\\ +\bar{g}(JY, Z)\bar{g}(JX, W) - \bar{g}(JX, Z)\bar{g}(JY, W)\\ +2\bar{g}(X, JY)\bar{g}(JZ, W)\end{array}\right]\\ &= \frac{c}{2}\bar{g}(X, JY)\bar{g}(JZ, W) \neq 0\end{aligned}$$

then (iv) holds.

$\Longleftarrow$) : Conversely, we suppose that (i), (ii), (iii) and (iv) are satisfied. From (i), $D = D_0 \oplus D_1$ is maximal invariant subspaces and $Rad(TM) = D_1 \oplus D_2$, while D_1 is an invariant distribution on TM, D_2 isn't invariant on TM with respect to J. Because of this, $J(D_2) \subset \Gamma(tr(TM))$. Furthermore, for $ltr(TM) = L_1 \perp L_2$ and $\xi_1 \in \Gamma(D_1)$, $N_1 \in \Gamma(L_1)$, since $\bar{g}(\xi_1, N_1) = 1$, then $J(L_1)$ is an invariant distribution, too. On the other hand, from (ii), for any $X, Y \in \Gamma(D_0)$, N_1, $N_2 \in \Gamma(ltr(TM))$, we get

$$\begin{aligned}\bar{g}(\bar{R}(X, Y)N_1, N_2) &= \frac{c}{4}[\bar{g}(Y, N_1)\bar{g}(X, N_2) - \bar{g}(X, N_1)\bar{g}(Y, N_2)\\ &\quad +\bar{g}(JY, N_1)\bar{g}(JX, N_2) - \bar{g}(JX, N_1)\bar{g}(JY, N_2)\\ &\quad +2\bar{g}(X, JY)\bar{g}(JN_1, N_2)]\\ &= 0.\end{aligned}$$

Thus, we derive $\frac{c}{2}g(X, JY)\bar{g}(JN_1, N_2) = 0$ for any $X, Y \in \Gamma(D_0)$, N_1, $N_2 \in \Gamma(ltr(TM))$. Since $c \neq 0$ and D_0 is non-degenerate we conclude that $\bar{g}(JN_1, N_2) = 0$ that is $J(ltr(TM)) \cap Rad(TM) = 0$. Thus, $J(L_2)$ isn't belong to $Rad(TM)$ or $ltr(TM)$. From this, it is clear that $J(D_2) \neq L_2$ and then $J(D_2) \subset S(TM^{\perp})$. On the other hand, for any $\xi_2 \in \Gamma(D_2)$ and $N_2 \in \Gamma(L_2)$, since $\bar{g}(\xi_2, N_2) = 1$, then $J(L_2)$ is a distribution on $S(TM^{\perp})$, too. Thus, (A) of the definition of screen generic transversal lightlike submanifold is satisfied. From (iii), we see that there exists a non-degenerate anti-invariant distribution $D^{'}$ on $S(TM)$ such that since $\bar{g}(JZ, V) \neq 0$, for any $Z, V \in \Gamma(D^{'})$, then

$$\Gamma(J(D^{'})) \subset \Gamma(D^{'}) \subset \Gamma(S(TM)). \tag{2.53}$$

On the other hand, from (iv), we have $\bar{g}(JZ, W) = -\bar{g}(Z, JW)$, for any $X, Y \in \Gamma(D_0)$, $Z \in \Gamma(D^{'})$, $W \in \Gamma(TM^{\perp})$, but $W \notin \Gamma(\mu)$. So, it is clear that

$$\Gamma(J(D^{'})) \subset \Gamma(S(TM^{\perp})). \tag{2.54}$$

Thus, from (2.53) and (2.54), we obtain neither $\Gamma(J(D^{'}))$ is in $\Gamma(D^{'})$ totally, nor $\Gamma(J(D^{'}))$ is in $\Gamma(S(TM^{\perp}))$ totally, which satisfies (B) of the definition of screen generic transversal lightlike submanifold. This proves our assertion. □

Definition 2.15. We say that M is a D-geodesic screen generic transversal lightlike submanifold if its second fundamental form h satisfies

$$h(X, Y) = 0, \quad \forall X, Y \in \Gamma(D). \tag{2.55}$$

It is easy to see that M is a D-geodesic screen generic transversal lightlike submanifold if

$$h^l(X,Y) = h^s(X,Y) = 0\,, \quad \forall X, Y \in \Gamma(D). \tag{2.56}$$

On the other hand, if h satisfies

$$h(X,Y) = 0\,, \quad \forall X \in \Gamma(D)\,, \ Y \in \Gamma(D^{\mu}) \tag{2.57}$$

then M is called a mixed geodesic screen generic transversal lightlike submanifold.

Proposition 2.16. *The distribution D of a screen generic transversal lightlike submanifold M of $\bar{M}$ is a totally geodesic foliation in $\bar{M}$ iff M is D-geodesic and D is parallel respect to ∇ on M.*

Proof. Suppose that D defines a totally geodesic foliation in $\bar{M}$, that is, $\bar{\nabla}_X Y \in \Gamma(D)$, for any $X, Y \in \Gamma(D)$. Then, we get

$$\bar{g}(\bar{\nabla}_X Y, Z) = \bar{g}(\bar{\nabla}_X Y, N_2) = \bar{g}(\bar{\nabla}_X Y, \xi) = \bar{g}(\bar{\nabla}_X Y, W) = 0$$

for any $Z \in \Gamma(D^{'})$, $N_2 \in \Gamma(L_2)$, $\xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Considering (1.7), we obtain

$$\begin{aligned} \bar{g}(\bar{\nabla}_X Y, \xi) &= \bar{g}(h^l(X,Y), \xi), \\ \bar{g}(\bar{\nabla}_X Y, W) &= \bar{g}(h^s(X,Y), W) \end{aligned}$$

then, it is clear that, for any $X, Y \in \Gamma(D)$, $h^l(X,Y) = h^s(X,Y) = 0$. In other words, M is D-geodesic and D is parallel respect to ∇ on M.
Conversely, we suppose that M is D-geodesic and D is parallel respect to ∇ on M. Since $h^l(X,Y) = h^s(X,Y) = 0$, for any $X, Y \in \Gamma(D)$, then $\bar{\nabla}_X Y \in \Gamma(TM)$. On the other hand, since D is parallel on M, using (1.7), we have $\bar{\nabla}_X Y \in \Gamma(D)$. This completes the proof. □

Theorem 2.17. *Let M be a screen generic transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then, M is mixed geodesic iff the followings are holded:*

$$\bar{g}(h^l(X,\Phi Z) + D^l(X,\omega Z), B\xi) = \bar{g}(h^s(X,\Phi Z) + \nabla^s_X \omega Z, C\xi), \tag{2.58}$$

$$\bar{g}(A_{\omega Z}X - \nabla_X \Phi Z, JW) = \bar{g}(h(X,\Phi Z) + \nabla^t_X \omega Z, JW), \tag{2.59}$$

for any $X \in \Gamma(D)$, $Z \in \Gamma(D^{\mu})$, $\xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^{\perp}))$.

Proof. Assume that M is mixed geodesic, then using (2.56), $\bar{g}(h^l(X,Z),\xi) = 0$ and $\bar{g}(h^s(X,Z),W) = 0$ for any $X \in \Gamma(D)$, $Z \in \Gamma(D^{\mu})$, $\xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Thus considering (1.5), (1.6), (1.7), (1.9), (1.21), (1.22), (2.11) and (2.13), we obtain

$$\begin{aligned} \bar{g}(h^l(X,Z),\xi) &= \bar{g}(h^l(X,\Phi Z) + h^s(X,\Phi Z) + \nabla^s_X \omega Z + D^l(X,\omega Z), J\xi), \\ \bar{g}(h^s(X,Z),W) &= \bar{g}(\nabla_X \Phi Z - A_{\omega Z}X, JW) + \bar{g}(h(X,\Phi Z) + \nabla^t_X \omega Z, JW) \end{aligned}$$

which proves our assertion. □

For any $Y \in \Gamma(TM)$, differentiating (2.11) and using (1.7), (1.9), (1.22), (2.11) and (2.13), we get

$$\begin{aligned} &\nabla_X \Phi Y + h^l(X,\Phi Y) + h^s(X,\Phi Y) - A_{\omega Y}X + \nabla^s_X \omega Y + D^l(X,\omega Y) \\ = \ &\Phi \nabla_X Y + \omega \nabla_X Y + Ch^l(X,Y) + Bh^s(X,Y) + Ch^s(X,Y). \end{aligned}$$

Taking tangential, lightlike transversal and screen transversal parts of this equation, we obtain

$$(\nabla_X \Phi)Y = A_{\omega Y}X + Bh(X,Y), \tag{2.60}$$

$$h^l(X,\Phi Y) = -D^l(X,\omega Y) + Ch^s(X,Y), \tag{2.61}$$

$$h^s(X,\Phi Y) = -\nabla^s_X \omega Y + \omega \nabla_X Y + Ch^l(X,Y) + Ch^s(X,Y). \tag{2.62}$$

Lemma 2.18. *Let M be a screen generic transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then M is mixed geodesic iff*

$$\nabla_X \Phi Z - A_{wZ}X \in \Gamma(D), \tag{2.63}$$

$$h^l(X, \Phi Z) = -D^l(X, wZ), \tag{2.64}$$

$$h^s(X, \Phi Z) + \nabla^s_X wZ \in \Gamma(J(D_2)) \tag{2.65}$$

for any $X \in \Gamma(D)$, $Z \in \Gamma(D^\mu)$.

Proof. Considering (1.7), (1.9), (1.22) and (2.11), we derive

$$\begin{aligned} h(X,Z) &= -J(\bar{\nabla}_X \Phi Z + wZ) - \nabla_X Z \\ &= -J(\nabla_X \Phi Z + h^l(X, \Phi Z) + h^s(X, \Phi Z) - A_{wZ}X + \nabla^s_X wZ \\ &\quad + D^l(X, wZ)) - \nabla_X Z \end{aligned}$$

for any $X \in \Gamma(D)$, $Z \in \Gamma(D^\mu)$. Considering (2.11) and (2.13) in the last equation and taking transversal part of resulting equation, we have

$$h(X,Z) = -w(\nabla_X \Phi Z - A_{wZ}X) - C(h^l(X, \Phi Z) + h^s(X, \Phi Z) + \nabla^s_X wZ + D^l(X, wZ)).$$

Hence, $h(X,Z) = 0$ iff (2.63), (2.64) and (2.65) are satisfied. □

Lemma 2.19. *Let M be a screen generic transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then, we have*

$$\nabla_X Z = \Phi A_{\omega Z}X - \Phi \nabla_X \Phi Z - Bh^s(X, \Phi Z) - B\nabla^s_X \omega Z$$

for any $X \in \Gamma(D_0)$, $Z \in \Gamma(D^\mu)$.

Proof. Using (1.7), (1.9), (1.21), (1.22), (2.11) and (2.13), we get

$$\begin{aligned} \bar{\nabla}_X Z &= -\Phi \nabla_X \Phi Z - w\nabla_X \Phi Z - Ch^l(X, \Phi Z) - Bh^s(X, \Phi Z) \\ &\quad -Ch^s(X, \Phi Z) + \Phi A_{wZ}X + wA_{wZ}X - B\nabla^s_X wZ \\ &\quad -C\nabla^s_X wZ - CD^l(X, wZ) \end{aligned}$$

for any $X \in \Gamma(D_0)$, $Z \in \Gamma(D^\mu)$. If we take tangential parts of the last equation, then we obtain

$$\nabla_X Z = \Phi A_{\omega Z}X - \Phi \nabla_X \Phi Z - Bh^s(X, \Phi Z) - B\nabla^s_X \omega Z$$

which proves our assertion. □

Theorem 2.20. *There exist no totally umbilical proper screen generic transversal lightlike submanifold of an indefinite complex space form $\bar{M}(c)$, $c \neq 0$.*

Proof. Suppose that M is a totally umbilical proper screen generic transversal lightlike submanifold of $\bar{M}(c)$, $c \neq 0$. Then, from (1.18) and (1.23) we get

$$\bar{R}(X, JX)Z = -\frac{c}{2} g(X,X) JZ \tag{2.66}$$

and

$$\bar{R}(X, JX)Z = (\nabla_X h^s)(JX, Z) - (\nabla_{JX} h^s)(X, Z), \tag{2.67}$$

for any $X \in \Gamma(D_0)$, $Z \in \Gamma(D^\mu)$. Since M is totally umbilical, from (1.20) we obtain

$$(\nabla_X h^s)(JX, Z) = -g(\nabla_X JX, Z)H^s - g(JX, \nabla_X Z)H^s. \tag{2.68}$$

We know that

$$g(JX, Z) = 0.$$

From the last equation, we get

$$(\bar{\nabla}_X g)(JX, Z) = g(\nabla_X JX, Z) + g(JX, \nabla_X Z) = 0. \tag{2.69}$$

Thus, using (2.68) and (2.69) we obtain

$$(\nabla_X h^s)(JX, Z) = 0. \tag{2.70}$$

Similarly, we have

$$(\nabla_{JX}h^s)(X,Z)=0. \tag{2.71}$$

Hence, using (2.66), (2.67), (2.70) and (2.71) we obtain

$$\frac{c}{2}g(X,X)JZ=0,$$

that is, $c=0$ is derived. This is a contradiction which completes the proof. □

Lemma 2.21. *Let M be a totally umbilical proper screen generic transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then $H^s\notin\Gamma(J(L_2)\perp\mu)$.*

Proof. Let M be a totally umbilical proper screen generic transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then, considering (2.62) we obtain

$$g(X,JY)H^s=w\nabla_XY+g(X,Y)CH^l+g(X,Y)CH^s$$

for any $X,Y\in\Gamma(D_0)$. If we get $X=JY$, then we get

$$g(Y,Y)H^s=\omega\nabla_{JY}Y.$$

Thus, we get $H^s\notin\Gamma(J(L_2)\perp\mu)$. This completes the proof. □

Theorem 2.22. *Let M be a totally umbilical screen generic transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then the induced connection ∇ is a metric connection.*

Proof. Assume that M is totally umbilical. Then, considering (2.61) we obtain

$$g(X,JY)H^l=g(X,Y)CH^s$$

for any $X,Y\in\Gamma(D_0)$. If we take $X=JY$ in the last equation then we get

$$g(Y,Y)H^l=0.$$

Since D_0 is non-degenerate, then $H^l=0$ that is, from (1.20) $h^l=0$. □

3. Minimal screen generic transversal lightlike submanifolds

Definition 3.1. We say that a lightlike submanifold M of a semi-Riemannian manifold $(\bar{M},\bar{g})$ is minimal if:

(i) $h^s=0$ on $Rad(TM)$ and

(ii) $trh=0$, where trace is written with respect to g restricted to $S(TM)$.

It has been proved in [2] that the above definition is independent of $S(TM)$ and $S(TM^{\perp})$, but it depends on $tr(TM)$.

Example 3.2. Let $\bar{M}=(\mathbb{R}^{16}_4,\bar{g})$ be a semi-Euclidean space, where $\bar{g}$ is signature and $(-,-,-,-,+,+,+,+,+,+,+,+,+,+,+,+)$ be the standard coordinate system of $\mathbb{R}^{16}_4$

$$(x_1,x_2,x_3,x_4,...,x_{15},x_{16}).$$

Consider the complex structure J defined by

$$\begin{aligned}J(x_1,x_2,...,x_{15},x_{16}) &= (-x_2,x_1,-x_4,x_3,-x_6,x_5,-x_8,x_7,-x_{10},x_9,\\ &\quad -x_{12},x_{11},-x_{15}\cos\alpha-x_{14}\sin\alpha,-x_{16}\cos\alpha+x_{13}\sin\alpha,\\ &\quad x_{13}\cos\alpha+x_{16}\sin\alpha,x_{14}\cos\alpha-x_{15}\sin\alpha).\end{aligned}$$

Let M be a submanifold of $\mathbb{R}^{16}_4$ given by

$$\begin{aligned}&x_1=u_1-u_2\sin\alpha,x_2=u_1\sin\alpha+u_2,x_3=u_3,x_4=0,x_5=u_3\sin\alpha,\\ &x_6=0,x_7=-u_2\cos\alpha,x_8=u_1\cos\alpha,x_9=u_3\cos\alpha,x_{10}=0,\\ &x_{11}=\sin u_4\cosh u_5,x_{12}=\cos u_4\sinh u_5,x_{13}=\cos u_6\sinh u_7,\\ &x_{14}=\sin u_6\cosh u_7,x_{15}=-\cos u_6\cosh u_7,x_{16}=\sin u_6\sinh u_7.\end{aligned}$$

Then, TM is spanned by

$$Z_1 = \partial x_1 + \sin\alpha\partial x_2 + \cos\alpha\partial x_8, \quad Z_2 = -\sin\alpha\partial x_1 + \partial x_2 - \cos\alpha\partial x_7,$$
$$Z_3 = \partial x_3 + \sin\alpha\partial x_5 + \cos\alpha\partial x_9, \quad Z_4 = \cos u_4\cosh u_5\partial x_{11} - \sin u_4\sinh u_5\partial x_{12},$$
$$Z_5 = \sin u_4\sinh u_5\partial x_{11} + \cos u_4\cosh u_5\partial x_{12},$$
$$Z_6 = -\sin u_6\sinh u_7\partial x_{13} + \cos u_6\cosh u_7\partial x_{14} + \sin u_6\cosh u_7\partial x_{15} + \cos u_6\sinh u_7\partial x_{16},$$
$$Z_7 = \cos u_6\cosh u_7\partial x_{13} + \sin u_6\sinh u_7\partial x_{14} - \cos u_6\sinh u_7\partial x_{15} + \sin u_6\cosh u_7\partial x_{16},$$

where $Rad(TM) = Span\{Z_1, Z_2, Z_3\}$ and $D_0 = Span\{Z_4, Z_5\}$. By direct calculation, we obtain that $ltr(TM)$ is spanned by

$$N_1 = \frac{1}{2}(-\partial x_1 - \sin\alpha\partial x_2 + \cos\alpha\partial x_8),\ N_2 = \frac{1}{2}(\sin\alpha\partial x_1 - \partial x_2 - \cos\alpha\partial x_7),$$
$$N_3 = \frac{1}{2}(-\partial x_3 + \sin\alpha\partial x_5 + \cos\partial x_9).$$

Furthermore, the screen transversal bundle is spanned by

$$W_1 = JZ_3 = \partial x_4 + \sin\alpha\partial x_6 + \cos\alpha\partial x_{10},\ W_2 = JN_3 = \frac{1}{2}(-\partial x_4 + \sin\alpha\partial x_6 + \cos\alpha\partial x_{10}),$$
$$W_3 = \sin u_6\cosh u_7\partial x_{13} + \cos u_6\sinh u_7\partial x_{14} + \sin u_6\sinh u_7\partial x_{15} - \cos u_6\cosh u_7\partial x_{16},$$
$$W_4 = \cos u_6\sinh u_7\partial x_{13} - \sin u_6\cosh u_7\partial x_{14} + \cos u_6\cosh u_7\partial x_{15} + \sin u_6\sinh u_7\partial x_{16},$$
$$W_5 = -\cos\alpha\partial x_5 + \sin\alpha\partial x_9,\ W_6 = -\cos\alpha\partial x_6 + \sin\alpha\partial x_{10}.$$

Since $JW_3 \neq W_4$, then it is easy to see that $\mu = Span\{W_5, W_6\}$ and

$$\begin{aligned} JZ_6 &= \sin\alpha Z_7 + \cos\alpha W_3\,, \\ JZ_7 &= -\sin\alpha Z_6 - \cos\alpha W_4. \end{aligned}$$

Then, $D^{'} = Span\{Z_6, Z_7\}$ and M is a screen generic transversal lightlike submanifold of $\mathbb{R}^{16}_4$.

On the other hand, by direct computations and using Gauss and Weingarten formulas, we get

$$\bar{\nabla}_{Z_i} Z_T = 0\,, \quad i = 1,2,3,4,5\,, \quad 1 \leq T \leq 7$$

and

$$h^l(Z_6, Z_6) = 0\,, \quad h^l(Z_7, Z_7) = 0,$$
$$h^s(Z_6, Z_6) = \frac{1}{\sinh^2 u_7 + \cosh^2 u_7} W_4,$$
$$h^s(Z_7, Z_7) = -\frac{1}{\sinh^2 u_7 + \cosh^2 u_7} W_4.$$

Hence, we get

$$h^s(X, Z_1) = h^s(X, Z_2) = h^s(X, Z_3) = 0$$

for any $X \in \Gamma(TM)$, that is, $h^s = 0$ on $Rad(TM)$ and

$$trace\,|_{S(TM)}\, h = 0.$$

Then, it is clear that M is not totally geodesic, but it is a minimal screen generic transversal lightlike submanifold of $\bar{M} = \mathbb{R}^{16}_4$.

Theorem 3.3. *Let M be a screen generic transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then M is minimal iff*

$$traceA^*_{\xi_q|S(TM)} = 0, traceA_{W_\beta|S(TM)} = 0 \tag{3.1}$$

$\bar{g}(Y, D^l(X, W)) = 0$, *for any* $X, Y, \xi_q \in \Gamma(Rad(TM))$ *and* $W_\beta, W \in \Gamma(S(TM^\perp))$, *where* $\dim(TM) = m$, $\dim(tr(TM)) = n$, $\dim(Rad(TM)) = r$.

Proof. Suppose that M is a screen generic transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then M is minimal iff $h^s = 0$ on $Rad(TM)$ and $traceh = 0$ on $S(TM)$, i.e.

$$\begin{aligned} traceh\mid_{S(TM)} &= traceh\mid_{D_0} + traceh\mid_{D'} \\ &= \sum_{i=1}^{a} h(Z_i, Z_i) + \sum_{j=1}^{b} h(U_j, U_j) \\ &= 0. \end{aligned}$$

If we choose an orthonormal basis of $S(TM)$ as $\{e_i\}_{i=1}^{m-r}$, then we get

$$\begin{aligned} traceh \mid_{S(TM)} &= traceh\mid_{D_0} + traceh\mid_{D'} \\ &= \sum_{i=1}^{2a} \epsilon_i h(e_i, e_i) + \sum_{j=1}^{b} \epsilon_j h(e_j, e_j). \end{aligned} \tag{3.2}$$

Now from (1.10), we have $h^s = 0$ on $Rad(TM)$ iff $\bar{g}(Y, D^l(X, W)) = 0$, for any $X, Y \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. If we choose an orthonormal basis of $S(TM)$ as $\{e_i\}_{i=1}^{m-r}$, then we obtain

$$\begin{aligned} traceh\mid_{S(TM)} &= \sum_{i=1}^{2a} \varepsilon_i \left[h^l(e_i, e_i) + h^s(e_i, e_i)\right] + \sum_{j=1}^{b} \epsilon_j \left[h^l(e_j, e_j) + h^s(e_j, e_j)\right] \\ &= \sum_{i=1}^{2a} \varepsilon_i [\frac{1}{r} \sum_{q=1}^{r} \bar{g}(h^l(e_i, e_i), \xi_q) N_q + \frac{1}{n-r} \sum_{\beta=1}^{n-r} \bar{g}(h^s(e_i, e_i), W_\beta) W_\beta] \\ &\quad + \sum_{j=1}^{b} \varepsilon_j [\frac{1}{r} \sum_{q=1}^{r} \bar{g}(h^l(e_j, e_j), \xi_q) N_q + \frac{1}{n-r} \sum_{\beta=1}^{n-r} \bar{g}(h^s(e_j, e_j), W_\beta) W_\beta]. \end{aligned} \tag{3.3}$$

From (1.10) and (1.15) in (3.3), we get

$$\begin{aligned} traceh\mid_{S(TM)} &= \sum_{i=1}^{2a} \varepsilon_i [\frac{1}{r} \sum_{q=1}^{r} \bar{g}(A^*_{\xi_q} e_i, e_i) N_q + \frac{1}{n-r} \sum_{\beta=1}^{n-r} g(A_{W_\beta} e_i, e_i) W_\beta] \\ &\quad + \sum_{k=1}^{c} \varepsilon_k [\frac{1}{r} \sum_{q=1}^{r} \bar{g}(A^*_{\xi_q} e_j, e_j) N_q + \frac{1}{n-r} \sum_{\beta=1}^{n-r} g(A_{W_\beta} e_j, e_j) W_\beta]. \end{aligned}$$

Therefore we obtain

$$traceh\mid_{S(TM)} = traceA^*_{\xi_k}\mid_{D_0 \oplus D'} + traceA_{W_\beta}\mid_{D_0 \oplus D'}.$$

Hence, we get

$$traceA^*_{\xi_k}\mid_{D_0 \oplus D'} = 0 \quad and \quad traceA_{W_\beta}\mid_{D_0 \oplus D'} = 0.$$

From the last equation the proof is completed. □

Acknowledgements

We would like to thank to the referees for their valuable comments and suggestions.

Author contributions. All the co-authors have contributed equally in all aspects of the preparation of this submission.

Conflict of interest statement. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Funding. This paper is supported by the Scientific and Technological Council of Türkiye (TÜBITAK) with project number 123F450.

Data availability. No data was used for the research described in the article.

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