Research Article BEU Fen Bilimleri Dergisi 15(2): 726-740 (2026) DOI: 10.17798/bitlisfen.1822685

BİTLİS EREN UNIVERSITY
FEN BİLİMLERİ DERGİSİ
ISSN: 2147-3129 / e-ISSN: 2147-3188

A NEW CAPUTO-FABRIZIO HYBRID METHOD FOR THE NUMERICAL SOLUTION OF CAPUTO-FABRIZIO FRACTIONAL EPIDEMIC MODEL

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Article Info

Received: *November 12, 2025*
Revised: *April 23, 2026*
Accepted: *May 5, 2026*

Keywords

Epidemic Model,
Caputo-Fabrizio Fractional Operator,
q-homotopy Analysis Transform Method,
Kushare Transform.

ABSTRACT

This paper presents a novel Caputo–Fabrizio hybrid numerical method for examining the dynamics of fractional epidemic models defined by memory effects and non-singular kernels. The proposed method combines the Caputo–Fabrizio fractional derivative with a hybrid transform framework, facilitating the derivation of accurate and convergent approximate solutions for nonlinear epidemic systems. The model's parameters are analyzed in relation to infection transmission and recovery dynamics, whereas the fractional order α regulates the magnitude of memory and genetic influences. Numerical simulations indicate that the novel hybrid method exhibits remarkable concordance with established results for integer-order scenarios, while showcasing enhanced precision, computing efficiency, and robustness in fractional contexts. The suggested method offers a versatile and effective analytical instrument for investigating real-world epidemic occurrences exhibiting long-term memory traits.

1. INTRODUCTION

Epidemiology is a fundamental field in biomedical and public health sciences, focused on the systematic study of the distribution, determinants, and dynamics of health-related illnesses, especially infectious and chronic diseases, within populations. Epidemiology utilizes observational, experimental, and mathematical frameworks to measure disease burden, identify risk factors, clarify causal mechanisms, and assess the efficacy of preventive or therapeutic interventions. The fundamental tenet of the discipline is population-level inference: through the examination of incidence and prevalence patterns in conjunction with demographic, environmental, behavioral, and genetic factors, epidemiologists develop etiological models that transcend individual pathology to uncover the structural and systemic influences on morbidity and mortality. In recent decades, the methodological framework of epidemiology has broadened to include cohort and case-control studies, survival models, and spatial-temporal clustering methods. Recently, mechanistic and statistical models based on dynamical systems theory—particularly fractional-order and stochastic epidemic models—have become prominent as they facilitate the simulation of contagion processes in heterogeneous populations characterized by memory-dependent and non-Markovian dynamics. Epidemiology functions at the essential intersection of biology and society, requiring both quantitative precision and ethical contextual awareness in addressing emerging diseases, antibiotic resistance, and global health disparities [1–5].

Concurrently, fractional calculus has arisen as a revolutionary mathematical framework that extends classical differential and integral operators to arbitrary real or complex orders. This nonlocal extension, with intellectual origins in Leibniz, Euler, and Liouville, and subsequently formalized by Riemann and Caputo, has demonstrated its significance in modeling physical and biological systems exhibiting memory effects, hereditary behavior, and anomalous diffusion processes—phenomena that conventional integer-order models cannot adequately address [6–8]. The Caputo derivative is frequently favored among numerous formulations due to its alignment with physically significant beginning conditions [9–

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10]. Recent years have witnessed substantial advancements due to the implementation of fractional operators with non-singular kernels, such the Caputo–Fabrizio and Atangana–Baleanu derivatives, which improve the physical interpretability and smoothness of fractional dynamics [11–12]. These operators encapsulate long-range temporal and spatial relationships, rendering fractional calculus essential for modeling viscoelastic materials, electrochemical reactions, control systems, and biological processes. Thus, fractional modeling enhances mathematical representations of intricate dynamical systems and redefines the epistemological limits of applied mathematics and theoretical physics.

Within this broader context, the Caputo–Fabrizio fractional derivative, proposed in 2015 by Michele Caputo and Mauro Fabrizio, represents a pivotal advancement designed to overcome the singular kernel limitations inherent in classical fractional operators. In contrast to the Riemann–Liouville or conventional Caputo derivatives, the Caputo–Fabrizio operator utilizes an exponential, non-singular kernel that diminishes gradually over time. This adjustment removes unphysical singularities at the origin, therefore providing enhanced numerical stability and physical realism in modeling systems influenced by fading memory. These qualities render the operator especially appropriate for applications in viscoelasticity, anomalous transport, and biological systems where smooth memory decay prevails. Moreover, its mathematical formulation guarantees well-defined initial conditions aligned with integer-order models, establishing a conceptual link between classical and fractional dynamics. Due to these benefits, the Caputo–Fabrizio derivative has emerged as a fundamental element in contemporary fractional modeling, harmonizing mathematical coherence with phenomenological precision [13].

In recent years, fractional-order mathematical models have been progressively utilized to examine diverse epidemiological and behavioral dynamics, including smoking behaviors, disease propagation, and diffusion phenomena. Numerous scholars have shown that fractional differential equations offer enhanced understanding of the persistence and long-term behavior of phenomena in comparison to classical equations [14–16]. Diverse analytical and numerical strategies have been presented to solve these complex equations, with the homotopy analysis method (HAM)—originally devised by Liao [17,18]—emerging as a notably versatile and potent approach. The core principle of HAM is creating a homotopy that gradually transforms an initial approximation into the precise solution, offering a clear method for regulating convergence. Expanding upon this basis, Prakash et al. [19] presented the q-homotopy analysis transform method (q-HATM) by amalgamating the Laplace transform with q-deformation principles, resulting in enhanced computational efficiency and analytical versatility.

In the present study, we are working out our own computational architecture by coupling the Caputo-Fabrizio fractional operator and Kushare transform of the q-homotopy analysis and the transformation of the q-homotopy analysis with the CFq-KHATM framework in a unified framework as opposed to those in [25], [26], [29] and [30] that, on the other hand, they mainly use q-homotopy analysis transform and that in [27] based on optimal q-homotopy analysis from the transform and without transformbased hybridization.

In addition, the present work utilizes a non-singular kernelbased Caputo-Fabrizio operator and combines it with a transform-homotopy transformation scheme and the same kind of modeling and solution perspective as [28] is present inside the conformable fractional derivative setting in, but the numerical approach is very different.

Although epidemic models with the Caputo-Fabrizio operator have been considered by and in the context of epidemics [31] and [32], it seems that only in the specific Caputo-Fabrizio-Kushare-q-homotopy hybrid framework such as our example is considered. The novelty of our work here will lay in the convergence of derivative choice, transform and deformation and iterative problem taking into account the hybrid method instead of just the use of the non-singular fractional derivative to compute the epidemic state nor a homotopy-based solver. Finally, numerical analysis is more comprehensive to perform a comparison with a standard algorithm for the calculation of fractions alone and provides a result which combines to describe convergence processes, stability, parametric sensitivity across fractional order, as well as the accuracy with the integer-order limit and as a result we have obtained approximate compartmental solutions.

Inspired by these advancements, the current study presents the Caputo–Fabrizio q-Kushare Homotopy Analysis Transform Method (CFq-KHATM) which a novel hybrid method that integrates the advantages

of the Caputo–Fabrizio fractional derivative, the q-homotopy analysis framework, and the Kushare transform. This synthesis offers a comprehensive numerical-analytical framework for addressing fractional epidemic models defined by non-singular memory kernels. The CFq-KHATM effectively addresses the limitations of conventional perturbation and discretization approaches by offering enhanced convergence control, improved accuracy, and strong parametric sensitivity while preserving the underlying fractional structure of the epidemic dynamics. Through this unified formulation, the study establishes a new computational paradigm capable of producing highly accurate approximate solutions to time-fractional epidemic systems. The suggested method enhances the theoretical basis of fractional epidemiological modeling and offers a realistic, adaptable framework for simulating intricate, memory-dependent biological systems regulated by generalized fractional operators.

The subsequent sections of this work are structured as follows: Section 2 delineates the essential mathematical prerequisites and articulates the principal definitions of fractional derivatives and the Kushare transform. Section 3 delineates the formulation of the proposed Caputo–Fabrizio q-Kushare homotopy analysis transform method (CFq-KHATM). Section 4 presents the numerical implementation and solutions of the Caputo–Fabrizio time-fractional epidemic model. Section 5 presents the results obtained and the concluding observations.

2. MATERIAL and METHOD

In the part, the basic definitions and theorems will be presented.

Definition 2.1. [20, 21] Suppose that $\mu \in (0,1]$ and ς be a continuously differentiable function. The CF fractional derivative (CFFD) of ς of order μ is given by

$$ {}^{CF}_{t_0}D^{\mu}_{0,t}[\varsigma(t)] = \frac{M(\mu)}{1-\mu}\int_0^t \varsigma'(\tau)\, exp\left[-\frac{\mu(t-\tau)}{1-\mu}\right]d\tau, \tag{1} $$

where $M(\mu)$ is the normalization function such that $M(0) = M(1) = 1$, and $\varsigma \in H^1(a,b), b > a$.

Definition 2.2. [22, 23] The Kushare transform (KT) for the function $\boldsymbol{\varsigma(t)}$ is defined by

$$ K[\,\varsigma(t)](\omega) = S(\omega) = \omega\int_0^{\infty} \exp\left(-t\omega^{\alpha}\right)\varsigma(\boldsymbol{t})dt\,, t \geq 0. \tag{2} $$

Definition 2.3. Let $\boldsymbol{0 < \mu \leq 1}, \boldsymbol{\varsigma}$: $[\boldsymbol{0}, \infty) \rightarrow \boldsymbol{R}$ be real valued function. The Caputo-Fabrizio fractional Kushare transform (CFFKT) of order $\boldsymbol{\mu}$ of $\boldsymbol{\varsigma(t)}$ is defined by

$$ {}^{CF}K_{\mu}\,[\varsigma(t)\,](\omega) = S_{\mu}(\omega) = \int_0^{\infty} \omega E_{\mu}[-\omega^{\alpha}, t]\varsigma(t)d_{\mu}t\,, t \geq 0, \tag{3} $$

where E_{μ} is the Mittag-Leffler function.

Definition 2.4. The CFFKT of the Caputo-Fabrizio fractional derivative (CFFD) for the function $\boldsymbol{\varsigma(t)}$ is described as

$$ {}^{CF}K_{\mu}\left\{{}^{CF}D^{\mu}_{t}[\varsigma(t)]\right\}(\omega) = \frac{1}{1-\mu+\mu\omega}\left(\omega^{\alpha+1\,CF}K_{\mu}\,[\varsigma(t)] - \omega^2\varsigma(0)\right) \tag{4} $$

2.1. The Novel Numerical Method

In this part, we present the Caputo-Fabrizio q-Kushare homotopy analysis transform method (CFq-KHATM), which combines with CFKT and q-homotopy analysis transform method.

Consider the Caputo-Fabrizio time-fractional order nonlinear partial differential equation (CFTFNPDE) to give the main idea of CFq-KHATM:

$$ {}^{CF}D_{\tau}^{\mu}u(\xi,\tau) + Au(\xi,\tau) + Hu(\xi,\tau) = \zeta(\xi,\tau), n-1 < \mu \leq n, \tag{5} $$

where A and H are linear and nonlinear operators, $\zeta(\xi,\tau)$ is a nonhomogeneous function, and ${}^{CF}D_{\tau}^{\mu}$ is CFD of order μ.

Step 1. By applying the CFKT to Eq. (5) and utilizing the initial condition (IC), Eq. (6) can be determined by

$$ \frac{1}{1-\mu+\mu\omega}\left(\omega^{\alpha+1\,CF}K_{\mu}\,[u(\xi,\tau)] - \omega^{2}u(\xi,0)\right) + {}^{CF}K_{\mu}\,[Au(\xi,\tau) + Hu(\xi,\tau) - \zeta(\xi,\tau)] = 0 \tag{6} $$

Rewriting the Eq. (6), Eq. (7) is obtained by

$$ {}^{CF}K_{\mu}\,[u(\xi,\tau)] - \omega^{1-\alpha}u(\xi,0) + \frac{1-\mu+\mu\omega}{\omega^{1+\alpha}}\,{}^{CF}K_{\mu}\,[Au(\xi,\tau) + Hu(\xi,\tau) - \zeta(\xi,\tau)] = 0 \tag{7} $$

Step 2. Via the HAM, the nonlinear operator of $\vec{u}(\xi,\tau;q)$ is described by

$$ \begin{aligned} N^{1}[\vec{u}(\xi,\tau;q)\,] &= {}^{CF}K_{\mu}\,[\vec{u}(\xi,\tau;q)\,] - \omega^{1-\alpha}\vec{u}(\xi,\tau;q)\,(0^{+}) \\ &+ \frac{1-\mu+\mu\omega}{\omega^{1+\alpha}}\left[{}^{CF}K_{\mu}\,[Au(\xi,\tau) + Hu(\xi,\tau) - \zeta(\xi,\tau)]\right], \end{aligned} \tag{8} $$

where $q\epsilon\left[0,\frac{1}{n}\right]$.

A homotopy is generated by

$$ (1-nq)\,{}^{CF}K_{\mu}\,[\vec{u}(\xi,\tau;q) - u_{0}(\xi,\tau)] = \hbar q\Theta^{*}(\xi,\tau)\,{}^{CF}K_{\mu}\,[\vec{u}(\xi,\tau;q)], \tag{9} $$

where ${}^{CF}K_{\mu}$ signifies CFKT and $\hbar \neq 0$ is an auxiliary parameter. The outcomes of Eq. (9) are determined for $q = 0$ and $q = \frac{1}{n}$ by

$$ \vec{u}(\xi,\tau;0) = u_{0}(\xi,\tau), \vec{u}\left(\xi,\tau;\frac{1}{n}\right) = u(\xi,\tau), \tag{10} $$

With q increasing from 0 to $1/n$, $\vec{u}(\xi,\tau;q)$ converges to the solution $u(\xi,\tau)$ from $u_{0}(\xi,\tau)$. When the Taylor theorem is applied to q, the result is found by

$$ \vec{u}(\xi,\tau;q) = u_{0}(\xi,\tau) + \sum_{i=1}^{\infty} u_{m}(\xi,\tau)q^{m}, \tag{11} $$

where,

$$ u_{m}(\xi,\tau) = \frac{1}{m!}\frac{\partial^{m}\vec{u}(\xi,\tau;q)}{\partial q^{m}}|_{q=0}. \tag{12} $$

For $u_{0}(\xi,\tau)$, n and $\hbar$, which are convenient, Equation (11) converges at $\mathrm{q} = \frac{1}{n}$.

By dividing by $m!$ and differentiating the zeroth-order deformation equation (11) m times with respect to q, it acquires the result for $q = 0$

$$ {}^{CF}K_{\mu}\,[u_{m}(\xi,\tau) - \hbar_{m}u_{m-1}(\xi,\tau)] = \hbar\Theta^{*}(\xi,\tau)\mathcal{R}_{1,m}(\vec{u}_{m-1}). \tag{13} $$

Step 3. When inverse CFKT (ICFKT) is utilized on Eq. (13), the outcome is obtained by

$$u_m(\xi,\tau) = \hbar_m u_{m-1}(\xi,\tau) + \hbar\left({}^{CF}K_\mu\right)^{-1}\left[\Theta^*(\xi,\tau)\mathcal{R}_{1,m}(\vec{u}_{m-1})\right], \tag{14}$$

where

$$\begin{aligned}\mathcal{R}_{1,m}(\vec{u}_{m-1}) &= {}^{CF}K_\mu\ [u_{m-1}(\xi,\tau)] - \omega^{1-\alpha}\left(1-\frac{\hbar_m}{n}\right)u_0(\xi,\tau)\\ &+\frac{1-\mu+\mu\omega}{\omega^{1+\alpha}}\left[{}^{CF}K_\mu\ [Au(\xi,\tau)+Hu(\xi,\tau)-\zeta(\xi,\tau)]\right]\end{aligned} \tag{15}$$

and

$$\hbar_m = \begin{cases}0, & m \le 1,\\ n, & m > 1\end{cases} \tag{16}$$

where, Θ^* is homotopy polynomial and it is identified by

$$\Theta^* = \frac{1}{m!}\frac{\partial^m \vec{u}(x,t;q)}{\partial q^m}|_{q=0}, \qquad \vec{u}(\xi,\tau;q) = \vec{u}_0 + q\vec{u}_1 + q^2\vec{u}_2 + \cdots \tag{17}$$

Utilizing Eqs. (14)-(15), one gets

$$\begin{aligned}u_m(\xi,\tau) &= (\hbar_m+\hbar)u_{m-1}(\xi,\tau) - \omega^{1-\alpha}\left(1-\frac{\hbar_m}{n}\right)u_0(\xi,\tau)\\ &+\hbar\left({}^{CF}K_\mu\right)^{-1}\left[\frac{1-\mu+\mu\omega}{\omega^{1+\alpha}}\,{}^{CF}K_\mu\ [Au_{m-1}(\xi,\tau)\Theta^*_{m-1}(\xi,\tau)-\zeta(\xi,\tau)]\right].\end{aligned} \tag{18}$$

Step 4. It is subsequently obtained through CFq-KHATM by

$$u(\xi,\tau) = u_0(\xi,\tau) + \sum_{m=1}^{\infty} u_m(\xi,\tau)\left(\frac{1}{n}\right)^m. \tag{19}$$

3. APPLICATION

In the part, the application of the CFq-KHATM to Caputo-Fabrizio time-fractional epidemic model (CFTFEM) is presented.

Example 3.1.

Let us analyze the nonlinear CFTFEM [24]

$${}^{CF}D_t^\mu A(t) = \beta - \gamma A(t)H(t) - \eta A(t),\ \ 0<\mu<1, \tag{20}$$

$${}^{CF}D_t^\mu C(t) = \gamma A(t)H(t) - (\varepsilon+\eta+\lambda)C(t),\ \ 0<\mu<1, \tag{21}$$

$${}^{CF}D_t^\mu H(t) = \lambda C(t) - (\rho+\eta)H(t), 0<\mu<1, \tag{22}$$

$${}^{CF}D_t^\mu L(t) = \rho H(t) + \varepsilon C(t) - \eta L(t),\ \ 0<\mu<1, \tag{23}$$

with ICs

$$\mathrm{A}(0) = 600, C(0) = 250, H(0) = 100, L(0) = 50. \tag{24}$$

Applying the CFKT to Eqs. (20-23) and using ICs, then it is found by

$$\frac{1}{1-\mu+\mu\omega}\left({}^{CF}K_{\mu}\,[A(t)]-\omega^{1-\alpha}A(0)\right)={}^{CF}K_{\mu}\,[\beta-\gamma A(t)H(t)-\eta A(t)], \tag{25}$$

$$\frac{1}{1-\mu+\mu\omega}\left({}^{CF}K_{\mu}\,[C(t)]-\omega^{1-\alpha}C(0)\right)={}^{CF}K_{\mu}\,[\gamma A(t)H(t)-(\varepsilon+\eta+\lambda)C(t)], \tag{26}$$

$$\frac{1}{1-\mu+\mu\omega}\left({}^{CF}K_{\mu}\,[H(t)]-\omega^{1-\alpha}H(0)\right)={}^{CF}K_{\mu}\,[\lambda C(t)-(\rho+\eta)H(t)], \tag{27}$$

$$\frac{1}{1-\mu+\mu\omega}\left({}^{CF}K_{\mu}\,[L(t)]-\omega^{1-\alpha}L(0)\right)={}^{CF}K_{\mu}\,[\rho H(t)+\varepsilon C(t)-\eta L(t)], \tag{28}$$

Rewriting the Eqs. (25-28), then it is obtained as

$${}^{CF}K_{\mu}\,[A(t)]-\omega^{1-\alpha}A(0)-\frac{1-\mu+\mu\omega}{\omega^{1+\alpha}}\,{}^{CF}K_{\mu}\,[\beta-\gamma A(t)H(t)-\eta A(t)]=0, \tag{29}$$

$${}^{CF}K_{\mu}\,[C(t)]-\omega^{1-\alpha}C(0)-\frac{1-\mu+\mu\omega}{\omega^{1+\alpha}}\,{}^{CF}K_{\mu}\,[\gamma A(t)H(t)-(\varepsilon+\eta+\lambda)C(t)]=0, \tag{30}$$

$${}^{CF}K_{\mu}\,[H(t)]-\omega^{1-\alpha}H(0)-\frac{1-\mu+\mu\omega}{\omega^{1+\alpha}}\,{}^{CF}K_{\mu}\,[\lambda C(t)-(\rho+\eta)H(t)]=0, \tag{31}$$

$${}^{CF}K_{\mu}\,[L(t)]-\omega^{1-\alpha}L(0)-\frac{1-\mu+\mu\omega}{\omega^{1+\alpha}}\,{}^{CF}K_{\mu}\,[\rho H(t)+\varepsilon C(t)-\eta L(t)]=0. \tag{32}$$

Via the homotopy analysis method, the nonlinear operator of $\phi(t;q)$, $\Psi(t;q),\Omega(t;q)$, $\varphi(t;q)$ are described by

$$N^{1}[\phi(t;q),\Psi(t;q),\Omega(t;q),\varphi(t;q)]={}^{CF}K_{\mu}\,[\phi(t;q)]-\omega^{1-\alpha}\phi(t;q)(0^{+})-\frac{1-\mu+\mu\omega}{\omega^{1+\alpha}}\,{}^{C}K_{\mu}\,[\beta-\gamma\phi\,\Omega-\eta\phi], \tag{33}$$

$$N^{2}[\phi(t;q),\Psi(t;q),\Omega(t;q),\varphi(t;q)]={}^{CF}K_{\mu}\,[\Psi(t;q)]-\omega^{1-\alpha}\Psi(t;q)(0^{+})-\frac{1-\mu+\mu\omega}{\omega^{1+\alpha}}\,{}^{CF}K_{\mu}\,[\gamma\phi\,\Omega-(\varepsilon+\eta+\lambda)\Psi], \tag{34}$$

$$N^{3}[\phi(t;q),\Psi(t;q),\Omega(t;q),\varphi(t;q)]={}^{CF}K_{\mu}\,[\,\Omega(t;q)]-\omega^{1-\alpha}\,\Omega(t;q)(0^{+})-\tfrac{1-\mu+\mu\omega}{\omega^{1+\alpha}}\,{}^{CF}K_{\mu}\,[\lambda\Psi-(\rho+\eta)\,\Omega], \tag{35}$$

$$N^{4}[\phi(t;q),\Psi(t;q),\Omega(t;q),\varphi(t;q)]={}^{CF}K_{\mu}\,[\varphi(t;q)]-\omega^{1-\alpha}\varphi(t;q)(0^{+})-\frac{1-\mu+\mu\omega}{\omega^{1+\alpha}}\,{}^{CF}K_{\mu}\,[\rho\Omega+\varepsilon\Psi-\eta\varphi], \tag{36}$$

where $q\epsilon\left[0,\frac{1}{n}\right]$.

Applying the proposed algorithm, the $m-th$ order deformation equation is described as

$$^{CF}K_{\mu}\ [A_m(t) - k_m A_{m-1}(t)] = h\mathcal{R}_{1,m}(\vec{A}_{m-1}). \tag{37}$$

$$^{CF}K_{\mu}\ [C_m(t) - k_m C_{m-1}(t)] = h\mathcal{R}_{2,m}(\vec{C}_{m-1}). \tag{38}$$

$$^{CF}K_{\mu}\ [H_m(t) - k_m H_{m-1}(t)] = h\mathcal{R}_{3,m}(\vec{H}_{m-1}). \tag{39}$$

$$^{CF}K_{\mu}\ [L_m(t) - k_m L_{m-1}(t)] = h\mathcal{R}_{4,m}(\vec{L}_{m-1}). \tag{40}$$

where,

$$\mathcal{R}_{1,m}(\vec{A}_{m-1}) = {}^{CF}K_{\mu}\ [A_{m-1}(t)] - \left(1 - \frac{k_m}{n}\right)\omega^{1-\alpha}A_0(t) - \frac{1-\mu+\mu\omega}{\omega^{1+\alpha}}\,{}^{CF}K_{\mu}\left[\beta - \gamma\left(\sum_{j=0}^{i} A_j(x,t)\,H_{i-j}(t)\right) - \eta A_{m-1}(t)\right], \tag{41}$$

$$\mathcal{R}_{2,m}(\vec{C}_{m-1}) = {}^{CF}K_{\mu}\ [C_{m-1}(t)] - \left(1 - \frac{k_m}{n}\right)\omega^{1-\alpha}C_0(t) - \frac{1-\mu+\mu\omega}{\omega^{1+\alpha}}\,{}^{CF}K_{\mu}\left[\gamma\left(\sum_{j=0}^{i} A_j(x,t)\,H_{i-j}(t)\right) - (\varepsilon+\eta+\lambda)C_{m-1}(t)\right], \tag{42}$$

$$\mathcal{R}_{3,m}(\vec{H}_{m-1}) = {}^{CF}K_{\mu}\ [H_{m-1}(t)] - \left(1 - \frac{k_m}{n}\right)\omega^{1-\alpha}H_0(t) - \frac{1-\mu+\mu\omega}{\omega^{1+\alpha}}\,{}^{CF}K_{\mu}\ [\lambda C_{m-1}(t) - (\rho+\eta)H_{m-1}(t)], \tag{43}$$

$$\mathcal{R}_{4,m}(L_{m-1}) = {}^{CF}K_{\mu}\ [L_{m-1}(t)] - \left(1 - \frac{k_m}{n}\right)\omega^{1-\alpha}L_0(t) - \frac{1-\mu+\mu\omega}{\omega^{1+\alpha}}\,{}^{CF}K_{\mu}\ [\rho H_{m-1}(t) + \varepsilon C_{m-1}(t) - \eta L_{m-1}(t)]. \tag{44}$$

Implementing the inverse CFKT (ICFKT) to Eqs. (37-40), then it is acquired by

$$A_m(t) = k_m A_{m-1}(t) + h\left({}^{CF}K_{\mu}\right)^{-1}\left[\mathcal{R}_{1,m}(\vec{A}_{m-1})\right], \tag{45}$$

$$C_m(t) = k_m C_{m-1}(t) + h\left({}^{CF}K_{\mu}\right)^{-1}\left[\mathcal{R}_{2,m}(\vec{C}_{m-1})\right], \tag{46}$$

$$H_m(t) = k_m H_{m-1}(t) + h\left({}^{CF}K_{\mu}\right)^{-1}\left[\mathcal{R}_{3,m}(\vec{H}_{m-1})\right], \tag{47}$$

$$L_m(t) = k_m L_{m-1}(t) + h\left({}^{CF}K_{\mu}\right)^{-1}\left[\mathcal{R}_{4,m}(\vec{L}_{m-1})\right]. \tag{48}$$

with ICs, it is obtained by

$$A_0(t) = 600, C_0(t) = 250, H_0(t) = 100, L_0(t) = 50 \tag{49}$$

Putting the values of $m = 1$ in the Eqs. (42-45), Eqs. (47-50) are found as

$$A_1(t) = -h(\beta - 60000\gamma - 600\eta)(1 - \mu + \mu t), \tag{50}$$

$$C_1(t) = -h[60000\gamma - 250(\varepsilon + \eta + \lambda)](1 - \mu + \mu t), \tag{51}$$

$$H_1(t) = -h[250\lambda - 100(\rho + \eta)](1 - \mu + \mu t), \tag{52}$$

$$L_1(t) = -h[100\rho + 250\varepsilon - 50\eta](1 - \mu + \mu t). \tag{53}$$

Other iterations are found similarly. Therefore, the CFq-KHATM solution of Eqs. (20-23) is found as

$$A(t) = 600 - h(\beta - 60000\gamma - 600\eta)(1 - \mu + \mu t), \tag{54}$$

$$C(t) = 250 - h[60000\gamma - 250(\varepsilon + \eta + \lambda)](1 - \mu + \mu t), \tag{55}$$

$$H(t) = 100 - h[250\lambda - 100(\rho + \eta)](1 - \mu + \mu t), \tag{56}$$

$$L(t) = 50 - h[100\rho + 250\varepsilon - 50\eta](1 - \mu + \mu t). \tag{57}$$

2D graphs for $A(t), C(t), H(t), L(t)$ solutions with CFq-KHATM by distinct μ values are shown in Figure 1. In Figure 1, the parameter values are taken as $\beta = 0.32, \gamma = 0.01, \eta = 0.2, \varepsilon = 0.25, \lambda = 0.01, \rho = 0.2$.

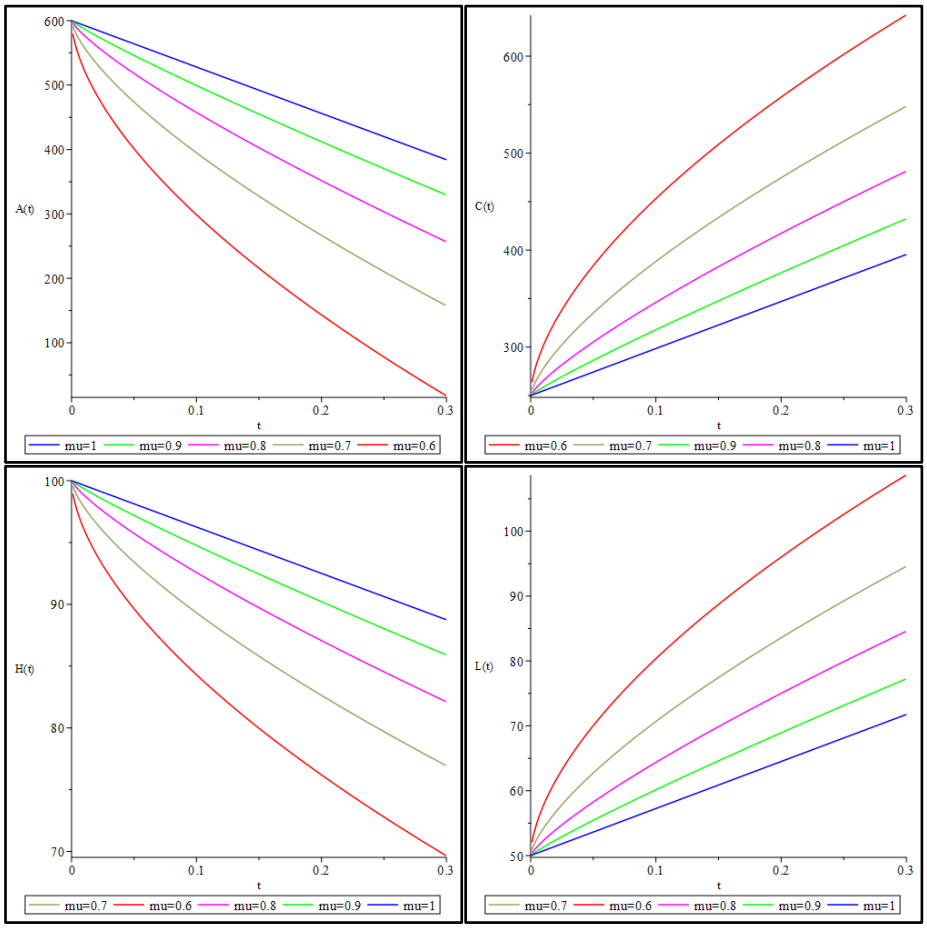


Figure 1. 2D plots of CFq-KHATM solutions of CFTFEM at $h = -1, n = 1$ *with distinct* μ.

ħ-Curves for $A(t), C(t), H(t), L(t)$ solutions with CFq-KHATM by distinct μ values are shown in Figures 2-5. In Figures 2-5, the parameter values are taken as $\beta = 0.32, \gamma = 0.01, \eta = 0.2, \varepsilon =$

$0.25, \lambda = 0.01, \rho = 0.2$. Viewing Figures 2-5 makes clear the importance of the auxiliary parameter h in the results obtained.

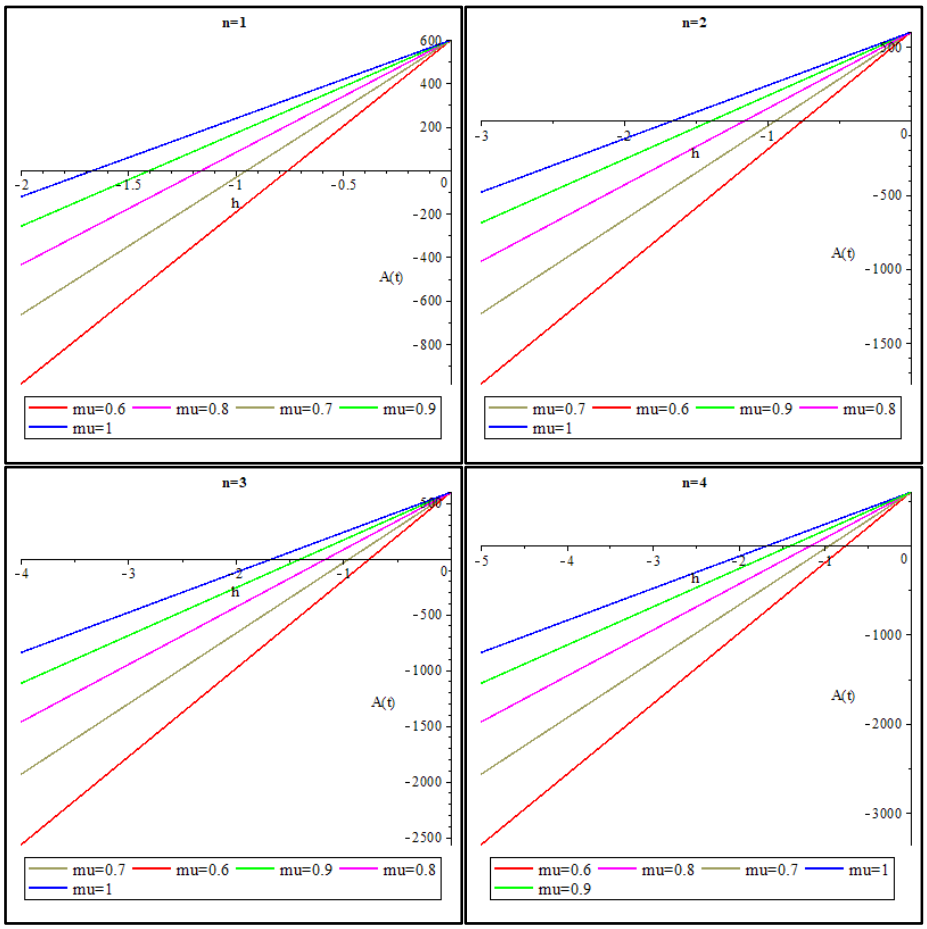


Figure 2. ħ-Curves for $A(t)$ of CFTFEM with different at $t = 0.5$ for distinct n.

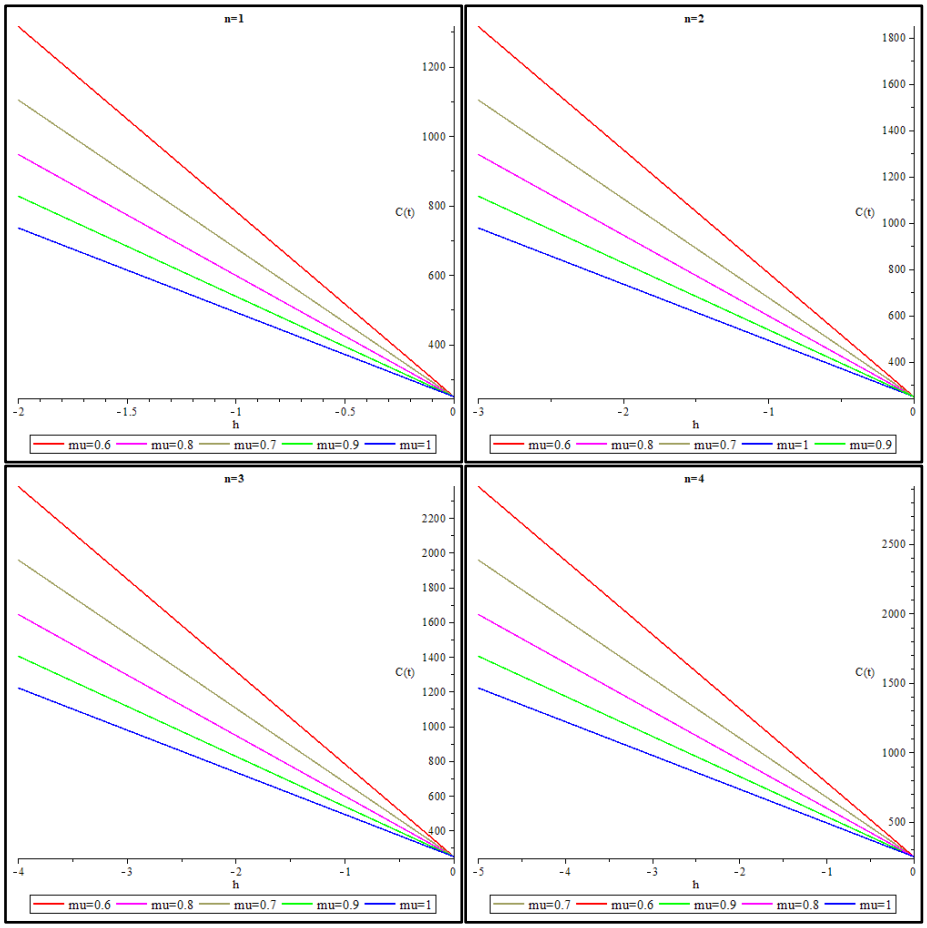


Figure 3. ħ-Curves for $C(t)$ *of CFTFEM with different at* $t = 0.5$ *for distinct* n.

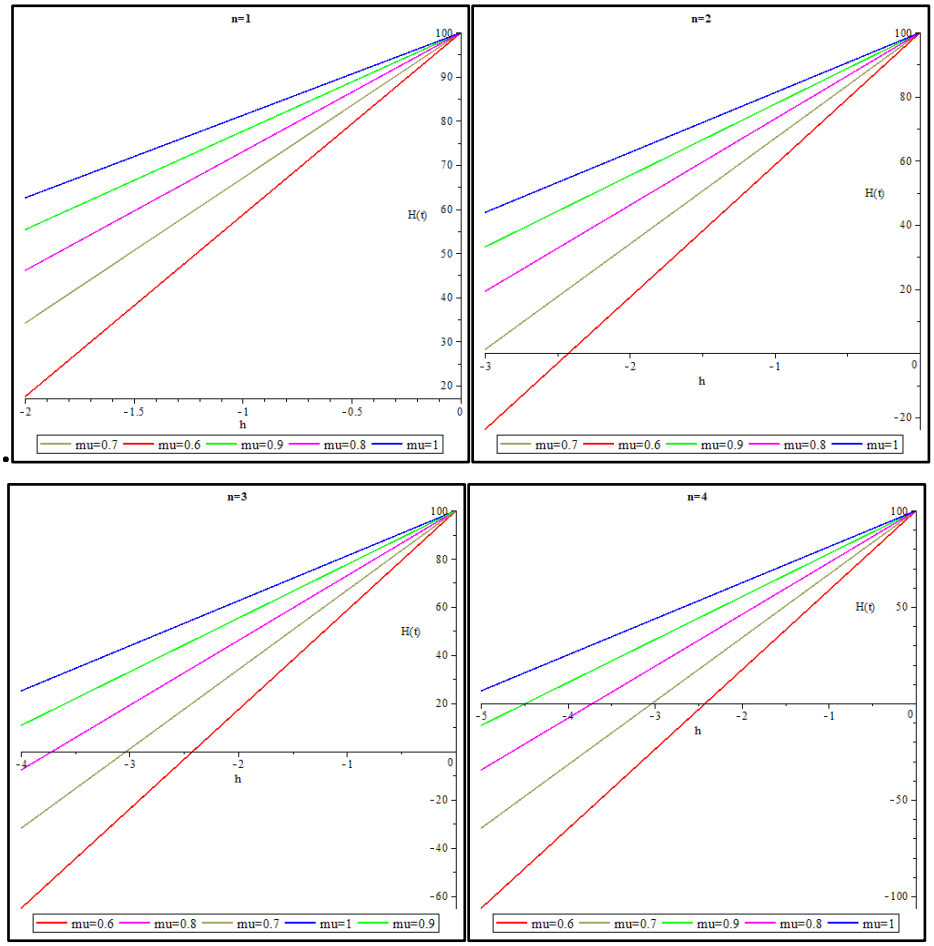


Figure 4. ħ-Curves for H(t) of CFTFEM with different at t=0.5 for distinct n.

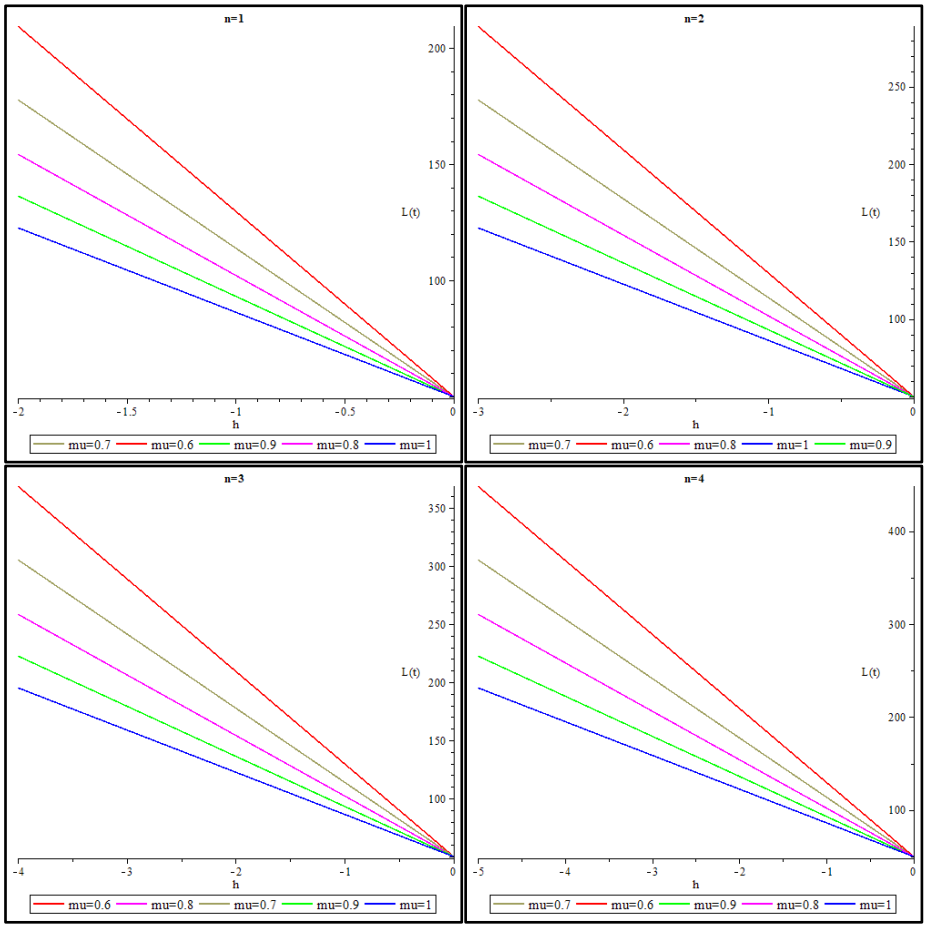


Figure 5. h-Curves for L(t) of CFTFEM with different at t=0.5 for distinct n.

4. RESULTS and DISCUSSION

The implementation of the Caputo-Fabrizio q-Kushare homotopy analysis transform method (CFq-KHATM) on the Caputo-Fabrizio time-fractional epidemic model (CFTFEM) demonstrates significant findings concerning both methodological efficacy and the epidemiological implications of the results. The numerical simulations demonstrate that the suggested method guarantees swift convergence and stability while maintaining accuracy throughout a broad spectrum of fractional orders. This validates the approach as a dependable alternative for evaluating nonlinear fractional epidemic systems that generally have significant memory effects. Figure 1 illustrates the model's solutions over several fractional orders. The fractional parameter μ markedly influences the paths of the susceptible, exposed, infected, and recovered compartments. At lower levels of μ, disease progression decelerates, resulting in a delayed peak and extended persistence of the infected population over time. This result demonstrates the impact of memory effects, wherein previous states of the system significantly affect the present dynamics. In contrast, when μ approaches one, the solutions align with the dynamics of the traditional SEIR model, resulting in more abrupt changes and expedited stabilization of the epidemic.

The findings further illustrate that incorporating fractional derivatives into the model facilitates the identification of concealed dynamics inadequately represented by integer-order models. The slow reduction of the infected population in fractional situations more accurately represents the enduring effects of infection and immunity in actual epidemics. Likewise, the gradual increase of the exposed class for lower fractional orders indicates a more accurate depiction of incubation and latency durations. These characteristics underscore the significance of fractional modeling in offering more adaptable and biologically pertinent representations of epidemic phenomena.

From a computational perspective, the CFq-KHATM demonstrates significant efficiency. The iterative method converges steadily without oscillations or divergence, even in the presence of highly nonlinear variables. Furthermore, the technique accommodates the nonlinear interactions between compartments

without necessitating stringent assumptions or approximations. This guarantees that the findings obtained are consistent with the original model while preserving computational feasibility. Collectively, these findings suggest that the suggested methodology not only improves the analytical examination of the epidemic model but also offers profound insights into the qualitative dynamics of disease transmission under fractional conditions. The technique therefore reconciles mathematical tractability with biological realism in epidemic studies.

5. CONCLUSION and SUGGESTIONS

This study presented a novel Caputo-Fabrizio q-Kushare homotopy analysis transform method (CFq-KHATM) and successfully applied it to the Caputo-Fabrizio time-fractional epidemic model (CFTFEM). The findings obtained validate that the method yields precise, stable, and swiftly converging solutions while maintaining the nonlinear integrity of the system. The analysis reveals that the fractional order parameter significantly affects the dynamics of the epidemic model: lower fractional values accentuate memory effects by extending the duration of infection, while higher values replicate classical integer-order behavior.

The results underscore the proposed method's capacity to include epidemiological characteristics, including delayed incubation, extended infection durations, and progressive recovery, which are challenging to depict in conventional integer-order models. The CFq-KHATM provides a dependable framework for investigating epidemic dynamics in a fractional context by integrating analytical tractability with computing efficiency.

This study demonstrates that the CFq-KHATM serves as both an efficient numerical instrument and a significant framework for enhancing understanding of epidemic processes. The methodology can be advanced and expanded to encompass more intricate epidemic frameworks, stochastic models, and additional biological systems where memory and genetic influences are vital.

Acknowledgment

We would like to thank the referees and editors for their valuable contributions.

Conflict of Interest Statement

There is no conflict of interest between the authors.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics

Artificial Intelligence (AI) Contribution Statement

This manuscript was entirely written, edited, analyzed, and prepared without the assistance of any artificial intelligence (AI) tools. All content, including text, data analysis, and figures, was solely generated by the authors.

Contributions of the Authors

A.A.: Conceptualization, Methodology, Validation, Formal Analysis, Writing-Original Draft Preparation,

H.A.: Data Curation, Writing-Reviewing and Editing, Visualization, and Investigation.

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