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On Topologies Induced by Order Relations on Dual Numbers

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Abstract. Inequalities are often utilized in different areas of mathematics to establish theorems. The presence of inequalities plays a crucial role in the foundations of these branches. In this study, we examine the properties of order relations in the dual number system, inspired by the characteristics of order relations defined on real numbers. With the help of these relations, topologies have been constructed in the space $\mathbb{D}^n$, and the geometric modeling of some elements of these topologies has been carried out.

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1. Introduction

W. K. Clifford (1845 – 1879) introduced dual numbers as a means to support his explorations in geometry. Clifford demonstrated that dual numbers form an algebra but not a field. This is due to the fact that only dual numbers with non-zero real part possess an inverse element [7]. Kotelnikov introduced this concept to mechanics in 1895, and since then, it has found diverse applications such as screw systems, modeling plane joint, iterative methods for displacement analysis of spatial mechanisms, and inertial force analysis. In the early part of the twentieth century, Kotelnikov made enhancements to the concept of dual vectors [23]. After W. K. Clifford, E. Study [34] applied dual numbers and dual vectors in his investigations with regard to line geometry and kinematics. He focused extensively on presenting directed lines through dual unit vectors and introduced the mapping, now bearing his name. This mapping establishes a direct correlation between the points on the dual unit sphere in $\mathbb{D}^3$ and the directed lines of Euclidean 3−space, ensuring a one-to-one correspondence between them. These numbers also hold significance in field theory [16]. The fascinating application of dual numbers in field theory is notably demonstrated in a series of articles authored by Wald et al. [39]. These numbers have modern application fields such as the curve design methods in computer-aided geometric design and computer modeling of rigid bodies, linkages, robots, mechanism design, modeling of the human body and dynamics [15, 24].

Inequalities lack a lengthy historical background as an academic field. As a mathematical concept, however, they were utilized by ancient mathematicians. For instance, Euclid employed phrases like ”falls short” or ”is in excess of” to signify that one area surpasses another [14, 18]. Several studies have been conducted on inequality and its historical

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evolution [4, 19, 27]. Inequalities serve as fundamental tools in various mathematical fields such as algebra, geometry, trigonometry, and contemporary calculus [14]. More precisely, the presence of inequalities is necessary to formulate the notation of metric, which stands as one of the fundamental structures of a space. The use of inequalities to order numbers has applications in a wide range of theoretical and practical fields [20].

Topology is a branch of mathematics focused on the properties of a geometric object preserved under continuous deformations like bending, crumpling, twisting and stretching. Topology emerged as a distinct mathematical discipline in the early twentieth century, though certain isolated findings can be traced back several centuries. Among these are specific questions in geometry that were explored by L. Euler. One of the first practical applications of topology is regarded as his 1736 paper on the Seven Bridges of Königsberg. Topology has remained a highly influential and dynamic field of research in modern mathematics for many years. Topology finds applications in various fields including physics, computer science, biology, robotic fiber art, puzzles and games. Furthermore, topology finds widespread application in numerous branches of mathematics such as differential equations, knot theory, dynamical systems, and Riemann surfaces in complex analysis. Topology is also applied to describe the structure of space-time in the universe, analyze various biological systems like nano-structures and molecules, and contribute to string theory in physics [1, 5, 6, 8, 9, 12, 13, 21, 25, 30, 32, 33, 38, 40]. In recent years, several studies have investigated the topological structures of bicomplex and bihyperbolic number systems [29, 31, 35, 36]. In this paper, we introduce different relations on the system of dual numbers and demonstrate that topologies can be constructed by means of these relations.

It is well known that the absolute value of a complex number and the norm of a complex vector are real numbers, whereas the absolute value of a dual number and the norm of a dual vector are dual numbers. It is evident that an order relation on dual numbers is necessary for conducting mathematical studies in dual space. This article is based on the relations $<_{\mathbb{D}}$ and $<_1$. The relation $<_{\mathbb{D}}$ is included in the articles [3, 11]. Inspired by the order relation discussed in the study by Akbar et al., the relation $<_1$ on dual numbers has been expressed and some basic properties of this relation are investigated in this article. In 2023, using the relation $<_{\mathbb{D}}$, we constituted the topology $\overline{\tau}$ on $\mathbb{D}^n$ [3]. In this study, a topology called $\overline{\tau}_1$ has been constructed on $\mathbb{D}^n$ using the relation $<_1$ and it follows that $\overline{\tau}_1 \subseteq \overline{\tau}$. Besides, we show that the space $(\mathbb{D}^n, \overline{\tau}_1)$ is the Hausdorff space. Thus, it is shown that different relations on dual numbers can be defined and topologies can be constructed with the help of these relations. For better understanding of our research, the geometric modeling for some sets of the topology $\overline{\tau}_1$ in $\mathbb{R}^3$ has been investigated.

2. Preliminaries

A dual number, denoted as $\overline{\omega}$, is represented by an ordered pair (ω, ω^*), where $\omega, \omega^* \in \mathbb{R}$. Consider that the set $\mathbb{R} \times \mathbb{R}$ is denoted as $\mathbb{D}$. An equality and two inner operations on $\mathbb{D} = \{(\omega, \omega^*) \mid \omega, \omega^* \in \mathbb{R}\}$ are described as below:

$$\begin{aligned} \text{Equality:} \quad & \overline{\omega} = \overline{\upsilon} \Leftrightarrow \omega = \upsilon \text{ and } \omega^* = \upsilon^*, \\ \text{Addition:} \quad & \overline{\omega} \oplus \overline{\upsilon} = (\omega + \upsilon, \omega^* + \upsilon^*), \\ \text{Multiplication:} \quad & \overline{\omega} \odot \overline{\upsilon} = (\omega\upsilon, \omega\upsilon^* + \omega^*\upsilon). \end{aligned}$$

Taking account of the equality and the two operators on $\mathbb{D}$ with a set of real numbers $\mathbb{R}$ identified as above, the set $\mathbb{D}$ is called the dual numbers system and the element $\overline{\omega} = (\omega, \omega^*)$ is called a dual number. In a dual number $\overline{\omega} = (\omega, \omega^*) \in \mathbb{D}$, the real number ω is called the real part of $\overline{\omega}$, and the real number ω^* is called the dual part of $\overline{\omega}$. The unit element of multiplication operation in $\mathbb{D}$ or real unit in $\mathbb{D}$ is expressed as the dual number $(1, 0) = \overline{1}$. The dual number $(0, 1)$ is symbolized as ε and the dual number $(0, 1) = \varepsilon$ is called dual unit. According to the definition of multiplication operation, it is evident that $\varepsilon^2 = 0$.

Let us examine an element $\overline{\omega}$ represented in the form of $\overline{\omega} = (\omega, 0)$. Then, the mapping $\zeta : \mathbb{D} \to \mathbb{R}$, $\zeta(\omega, 0) = \omega$ is an isomorphism. Hence, we can write

$$\begin{aligned} \overline{\omega} &= (\omega, \omega^*) \\ &= (\omega, 0) \oplus (0, \omega^*) \\ &= (\omega, 0) \oplus (0, 1) \odot (\omega^*, 0) \\ &= \omega \oplus \varepsilon \odot \omega^*. \end{aligned}$$

For convenience, throughout this paper, we will use + and · instead of $\oplus$ and $\odot$, respectively. With these conventions, the set of all dual numbers is expressed as

$$\mathbb{D} = \left\{\overline{\omega} = \omega + \varepsilon\omega^* \mid \omega, \omega^* \in \mathbb{R}, \varepsilon^2 = 0\right\}.$$

The set $\mathbb{D}$ constitutes a commutative ring with unity under the operations

$$(\omega + \varepsilon\omega^*) + (\upsilon + \varepsilon\upsilon^*) = (\omega + \upsilon) + \varepsilon(\omega^* + \upsilon^*)$$

and

$$(\omega + \varepsilon\omega^*) \cdot (\upsilon + \varepsilon\upsilon^*) = \omega\upsilon + \varepsilon(\omega\upsilon^* + \omega^*\upsilon).$$

The dual number $\overline{\omega} = \omega + \varepsilon\omega^*$ divided by the dual number $\overline{\upsilon} = \upsilon + \varepsilon\upsilon^*$ satisfying the condition $\upsilon \neq 0$ can be identified by

$$\frac{\overline{\omega}}{\overline{\upsilon}} = \frac{\omega + \varepsilon\omega^*}{\upsilon + \varepsilon\upsilon^*} = \frac{\omega}{\upsilon} + \varepsilon\left(\frac{\omega^*\upsilon - \omega\upsilon^*}{\upsilon^2}\right)$$

[22, 37].

The set of dual vectors on $\mathbb{D}^n$ is constituted as follows:

$$\mathbb{D}^n = \left\{\overrightarrow{\overline{\omega}} = (\overline{\omega}_1, ..., \overline{\omega}_n) \mid \overline{\omega}_i \in \mathbb{D}, i = 1, ..., n\right\}.$$

These vectors on $\mathbb{D}^n$ can be expressed in the form $\overrightarrow{\overline{\omega}} = \left(\overrightarrow{\omega}, \overrightarrow{\omega}^*\right) = \overrightarrow{\omega} + \varepsilon\overrightarrow{\omega}^*$, where $\overrightarrow{\omega}$ and $\overrightarrow{\omega}^*$ are the vectors on $\mathbb{R}^n$. Consider that $\overrightarrow{\overline{\omega}} = \overrightarrow{\omega} + \varepsilon\overrightarrow{\omega}^*$ and $\overrightarrow{\overline{\upsilon}} = \overrightarrow{\upsilon} + \varepsilon\overrightarrow{\upsilon}^*$ are dual vectors on $\mathbb{D}^n$ and $\overline{\lambda} = \lambda + \varepsilon\lambda^*$ is a dual number. The set $\mathbb{D}^n$ is a module over the ring $\mathbb{D}$ according to the operators

$$\overrightarrow{\overline{\omega}} + \overrightarrow{\overline{\upsilon}} = (\overline{\omega}_1 + \overline{\upsilon}_1, ..., \overline{\omega}_n + \overline{\upsilon}_n)$$

and

$$\overline{\lambda}.\overrightarrow{\overline{\omega}} = \lambda\overrightarrow{\omega} + \varepsilon\left(\lambda\overrightarrow{\omega}^* + \lambda^*\overrightarrow{\omega}\right).$$

This module is called $\mathbb{D}-$module or n-dimensional dual space.

Observe that $\overline{x} = x + \varepsilon x^*$ is a dual variable. A dual variable function $\overline{\zeta} : \mathbb{D} \to \mathbb{D}$ is described as below:

$$\overline{\zeta}(\overline{x}) = \zeta(x, x^*) + \varepsilon\zeta^0(x, x^*),$$

where ζ and ζ^0 are real functions with two real variables x and x^*. Dimentberg [10] explored the characteristics of dual functions. He showed that the analytic (differentiable) conditions of the dual functions are

$$\frac{\partial\zeta}{\partial x^*} = 0 \text{ and } \frac{\partial\zeta^0}{\partial x^*} = \frac{\partial\zeta}{\partial x}.$$

With these conventions, the general notation of dual analytic functions is obtained as below:

$$\overline{\zeta}(\overline{x}) = \zeta(x) + \varepsilon\left(x^*\zeta'(x) + \widetilde{\zeta}(x)\right),$$

where $\widetilde{\zeta}$ is an arbitrary function of the real part of a dual variable. The derivative of the dual analytic function $\overline{\zeta}$ with respect to $\overline{x}$ is

$$\frac{d\overline{\zeta}}{d\overline{x}} = \zeta'(x) + \varepsilon\left(x^*\zeta''(x) + \widetilde{\zeta}'(x)\right).$$

This definition enables the expression of several well-known dual functions as below [10, 26]:

$$\sin(x + \varepsilon x^*) = \sin x + \varepsilon x^* \cos x$$

$$\cos(x + \varepsilon x^*) = \cos x - \varepsilon x^* \sin x$$

$$\sqrt[n]{x + \varepsilon x^*} = \sqrt[n]{x} + \varepsilon\frac{x^*}{n\sqrt[n]{x^{n-1}}}, \; x \neq 0.$$

Theorem 2.1 ([10]). *Assume that $\overline{x} = x + \varepsilon x^*$ is a dual variable. For $n \in \mathbb{N}$,*

$$\overline{x}^n = x^n + \varepsilon x^* n x^{n-1}.$$

Definition 2.2 ([28]). (Dictionary order relation) Given two words, they are ordered by comparing their first letters according to their positions in the alphabet. In case the first letters are identical, the comparison proceeds to their second letters for ordering .

Definition 2.3 ([28]). Consider that A and B are two sets with order relations $<_A$ and $<_B$, respectively. For (a_1, b_1) and $(a_2, b_2) \in A \times B = \{(a, b) \mid a \in A,\ b \in B\}$, the relation $(a_1, b_1) < (a_2, b_2)$ is described as below:

(i) One compares the first components of these pairs and they must be $a_1 <_A a_2$,

(ii) In case their first components are the same, then one compares their second components and they must be $b_1 <_B b_2$.

Inspired by this definition, order relation among dual numbers can be derived.

Let $\overline{\omega} = \omega + \varepsilon\omega^*$ and $\overline{\upsilon} = \upsilon + \varepsilon\upsilon^*$ be dual numbers. The relation $\overline{\omega} <_{\mathbb{D}} \overline{\upsilon}$ (resp. $\overline{\omega} \leq_{\mathbb{D}} \overline{\upsilon}$) between these dual numbers is as below:

1) Firstly, the real parts of these dual numbers are compared and must be $\omega < \upsilon$ (resp. $\omega \leq \upsilon$).

2) If the real parts of these dual numbers are the same, their dual parts are compared and must be $\omega^* < \upsilon^*$ (resp. $\omega^* \leq \upsilon^*$) [3, 11].

Corollary 2.4 ([3, 11]). *Let $\overline{\omega} = \omega + \varepsilon\omega^*$ and $\overline{\upsilon} = \upsilon + \varepsilon\upsilon^*$ be dual numbers. One can infer that there are the following relations:*

$$\overline{\omega} <_{\mathbb{D}} \overline{\upsilon} \Leftrightarrow \omega < \upsilon \text{ or } (\omega = \upsilon \text{ and } \omega^* < \upsilon^*)$$

and

$$\overline{\omega} \leq_{\mathbb{D}} \overline{\upsilon} \Leftrightarrow \omega < \upsilon \text{ or } (\omega = \upsilon \text{ and } \omega^* \leq \upsilon^*).$$

Definition 2.5. If X is a set, a basis for a topology on X is a collection β of subsets of X such that

(i) For each $x \in X$, there is at least one basis element B containing x.

(ii) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subseteq B_1 \cap B_2$.

If β satisfies these two conditions, then the topology τ generated by β is identified [28].

3. Dual Inequalities and Topologies

As is well known, the absolute value of a dual number $\overline{a} = a + \varepsilon a^*$ is of the form

$$|\overline{a}|_{\mathbb{D}} = \sqrt{\overline{a}^2} = \begin{cases} \overline{0} & a = 0, \\ |a| + \varepsilon a^* \dfrac{a}{|a|} & a \neq 0. \end{cases}$$

In [3], all relations between the dual absolute value function $|\cdot|_{\mathbb{D}}$ and the relation $<_{\mathbb{D}}$ have been investigated in detail. In this study, we will give a different definition of absolute value on dual numbers by utilizing the definition of dual-valued metric we have previously obtained [3]. After then, we will define the relation $<_1$ on the dual number system, examine its properties and show that bases and topologies can be constructed with the help of this relation.

Definition 3.1 ([3]). Consider that $\overline{\omega} = \omega + \varepsilon\omega^*$ be a dual number. The absolute value of dual number $\overline{\omega}$ is

$$|\overline{\omega}|_1 = |\omega| + \varepsilon\,|\omega^*|,$$

where the notation $|\cdot|$ is absolute value on $\mathbb{R}$.

This definition is different from the known definition of absolute value. However, in both definitions, the absolute value of a dual number is defined to be a dual number. Firstly, we will examine the relations between this Definition 3.1 and the relation $\leq_{\mathbb{D}}$.

Proposition 3.2. *Observe that $\overline{\omega} = \omega + \varepsilon\omega^*$ and $\overline{\upsilon} = \upsilon + \varepsilon\upsilon^*$ are two dual numbers. There exist the following properties.*

(i) $|-\overline{\omega}|_1 = |\overline{\omega}|_1$.

(ii) $|\overline{\omega}\overline{\upsilon}|_1 \leq_{\mathbb{D}} |\overline{\omega}|_1\,|\overline{\upsilon}|_1$.

(iii) $\left|\dfrac{\overline{\omega}}{\overline{\upsilon}}\right|_1 \geq_{\mathbb{D}} \dfrac{|\overline{\omega}|_1}{|\overline{\upsilon}|_1}$ *for* $\upsilon \neq 0$.

Proof. (*i*) From the Definition 3.1, we can write

$$|-\overline{\omega}|_1 = |-\omega| + \varepsilon\,|-\omega^*| = |\omega| + \varepsilon\,|\omega^*| = |\overline{\omega}|_1 .$$

(*ii*) From the Definition 3.1, it is possible to say

$$\begin{aligned}
|\overline{\omega\upsilon}|_1 &= |(\omega + \varepsilon\omega^*)\,(\upsilon + \varepsilon\upsilon^*)|_1 \\
&= |\omega\upsilon + \varepsilon\,(\omega\upsilon^* + \omega^*\upsilon)|_1 \\
&= |\omega\upsilon| + \varepsilon\,|\omega\upsilon^* + \omega^*\upsilon| \\
&\leq_{\mathbb{D}} |\omega\upsilon| + \varepsilon\,(|\omega\upsilon^*| + |\omega^*\upsilon|) \\
&= |\omega|\,|\upsilon| + \varepsilon\,(|\omega|\,|\upsilon^*| + |\omega^*|\,|\upsilon|) \\
&= (|\omega| + \varepsilon\,|\omega^*|)\,(|\upsilon| + \varepsilon\,|\upsilon^*|) \\
&= |\overline{\omega}|_1\,|\overline{\upsilon}|_1 .
\end{aligned}$$

(*iii*) From the Definition 3.1, we have

$$\begin{aligned}
\left|\frac{\overline{\omega}}{\overline{\upsilon}}\right|_1 &= \left|\frac{\omega + \varepsilon\omega^*}{\upsilon + \varepsilon\upsilon^*}\right|_1 \\
&= \left|\frac{\omega}{\upsilon} + \varepsilon\left(\frac{\omega^*}{\upsilon} - \frac{\omega\upsilon^*}{\upsilon^2}\right)\right|_1 \\
&= \left|\frac{\omega}{\upsilon}\right| + \varepsilon\left|\frac{\omega^*}{\upsilon} - \frac{\omega\upsilon^*}{\upsilon^2}\right| \\
&\geq_{\mathbb{D}} \frac{|\omega|}{|\upsilon|} + \varepsilon\left|\left|\frac{\omega^*}{\upsilon}\right| - \left|\frac{\omega\upsilon^*}{\upsilon^2}\right|\right| \\
&= \frac{|\omega|}{|\upsilon|} + \varepsilon\left|\frac{|\omega^*|}{|\upsilon|} - \frac{|\omega|\,|\upsilon^*|}{|\upsilon|^2}\right| \\
&\geq_{\mathbb{D}} \frac{|\omega| + \varepsilon\,|\omega^*|}{|\upsilon| + \varepsilon\,|\upsilon^*|} \\
&= \frac{|\overline{\omega}|_1}{|\overline{\upsilon}|_1}.
\end{aligned}$$

□

Now, in Definition 3.3, we introduce a new classification of the dual number system. This classification will be used in the statements and proofs of the propositions in this paper.

Definition 3.3. Let $\overline{\omega} = \omega + \varepsilon\omega^*$ be a dual number. The sets

$$\begin{aligned}
\mathbb{D}^{+\prime} &= \{\overline{\omega} = \omega + \varepsilon\omega^* \in \mathbb{D} \mid \omega > 0, \omega^* > 0\}, \\
\mathbb{D}^{+\prime\prime} &= \{\overline{\omega} = \omega + \varepsilon\omega^* \in \mathbb{D} \mid \omega > 0, \omega^* < 0\}, \\
\mathbb{D}^{-\prime} &= \{\overline{\omega} = \omega + \varepsilon\omega^* \in \mathbb{D} \mid \omega < 0, \omega^* < 0\}, \\
\mathbb{D}^{-\prime\prime} &= \{\overline{\omega} = \omega + \varepsilon\omega^* \in \mathbb{D} \mid \omega < 0, \omega^* > 0\}
\end{aligned}$$

are called complete dual positive, semi dual positive, complete dual negative and semi dual negative numbers, respectively.

Proposition 3.4. *For $\overline{\omega} = \omega + \varepsilon\omega^* \in \mathbb{D}$ and $n \in \mathbb{N}$, the following properties are satisfied.*
(i) $\left|\overline{\omega}^n\right|_1 = |\overline{\omega}|_1^n$.
(ii) $\left|\overline{\omega}^{-n}\right|_1 \geq_{\mathbb{D}} |\overline{\omega}|_1^{-n}$ *for* $\omega \neq 0$.

Proof. (*i*) By the definition of the notation $|.|_1$, in case of $\omega = 0$, it is clear that $\left|\overline{\omega}^n\right|_1 = |\overline{\omega}|_1^n$. Assume that $\omega \neq 0$. Then, we have

$$\begin{aligned}
|\overline{\omega}|_1^n &= (|\omega| + \varepsilon\,|\omega^*|)^n \\
&= |\omega|^n + \varepsilon n\,|\omega^*|\,|\omega|^{n-1} \\
&= |\omega^n| + \varepsilon n\,|\omega^*|\left|\omega^{n-1}\right| \\
&= |\omega^n| + \varepsilon\left|n\omega^*\omega^{n-1}\right| \\
&= \left|\omega^n + \varepsilon n\omega^*\omega^{n-1}\right|_1 \\
&= \left|\overline{\omega}^n\right|_1 .
\end{aligned}$$

(*ii*) Using the expression (*iii*) of the Proposition 3.2 and the expression (*i*) of the Proposition 3.4, it is possible to write

$$\begin{aligned}
\left|\overline{\omega^{-n}}\right|_1 &= \left|\frac{1}{\overline{\omega}^n}\right|_1 \\
&\geq_{\mathbb{D}} \frac{1}{\left|\overline{\omega}^n\right|_1} \\
&= \frac{1}{|\overline{\omega}|_1^n} \\
&= |\overline{\omega}|_1^{-n} .
\end{aligned}$$

□

Now, inspired by Definition 3.1, we present the following proposition. This Proposition will provide a perspective on defining a metric in the dual space and on deriving a basis and a topology from this metric. For further details, we refer to [3].

Proposition 3.5. *Given the function* $\|\cdot\|_1 : \mathbb{D}^n \to \mathbb{D}$, $\left\|\overline{\varrho}_1\right\|_1 = \|\varrho_1\| + \varepsilon\left\|\varrho_1^*\right\|$, *where* $\overline{\varrho}_1 = \varrho_1 + \varepsilon\varrho_1^*$ *and* $\varrho_1, \varrho_1^* \in \mathbb{R}^n$, *the following properties are satisfied.*
(i) $\left\|\overline{\varrho}_1\right\|_1 \geq_{\mathbb{D}} \overline{0}$ *for all* $\overline{\varrho}_1 \in \mathbb{D}^n$*; equality holds if and only if* $\overline{\varrho}_1 = \overline{0}$.
(ii) $\left\|\overline{\lambda}\overline{\varrho}_1\right\|_1 \leq_{\mathbb{D}} \left|\overline{\lambda}\right|_1 \left\|\overline{\varrho}_1\right\|_1$ *for all* $\overline{\varrho} \in \mathbb{D}^n$ *and* $\overline{\lambda} \in \mathbb{D}$. *Specially, in case of* $\overline{\lambda} = \lambda \in \mathbb{R}$, *then* $\left\|\overline{\lambda}\overline{\varrho}_1\right\|_1 = \left|\overline{\lambda}\right|_1 \left\|\overline{\varrho}_1\right\|_1$.
(iii) $\left\|\overline{\varrho}_1 + \overline{\varrho}_2\right\|_1 \leq_{\mathbb{D}} \left\|\overline{\varrho}_1\right\|_1 + \left\|\overline{\varrho}_2\right\|_1$ *for all* $\overline{\varrho}_1, \overline{\varrho}_2 \in \mathbb{D}^n$.

Proof. (*i*) Using the properties of the norm function in Euclidean space, it is clear that $\|\varrho_1\| \geq 0$ and $\left\|\varrho_1^*\right\| \geq 0$. From the partial order relation $\leq_{\mathbb{D}}$, we can write $\left\|\overline{\varrho}_1\right\|_1 = \|\varrho_1\| + \varepsilon\left\|\varrho_1^*\right\| \geq_{\mathbb{D}} \overline{0}$. Besides, it is easy to see that the equality $\left\|\overline{\varrho}_1\right\|_1 = \overline{0}$ is equivalent to $\overline{\varrho}_1 = \overline{0}$.

(*ii*) From the definition of the relation $\leq_{\mathbb{D}}$, we obtain

$$\begin{aligned}
\left\|\overline{\lambda}\overline{\varrho}_1\right\|_1 &= \left\|\lambda\varrho_1 + \varepsilon\left(\lambda\varrho_1^* + \lambda^*\varrho_1\right)\right\|_1 \\
&= \|\lambda\varrho_1\| + \varepsilon\left\|\lambda\varrho_1^* + \lambda_1^*\varrho_1\right\| \\
&= |\lambda|\,\|\varrho_1\| + \varepsilon\left\|\lambda\varrho_1^* + \lambda_1^*\varrho_1\right\| \\
&\leq_{\mathbb{D}} |\lambda|\,\|\varrho_1\| + \varepsilon\left(\left\|\lambda\varrho_1^*\right\| + \left\|\lambda_1^*\varrho_1\right\|\right) \\
&= |\lambda|\,\|\varrho_1\| + \varepsilon\left(|\lambda|\left\|\varrho_1^*\right\| + |\lambda^*|\,\|\varrho_1\|\right) \\
&= \left(|\lambda| + \varepsilon\,|\lambda^*|\right)\left(\|\varrho_1\| + \varepsilon\left\|\varrho_1^*\right\|\right) \\
&= \left|\overline{\lambda}\right|_1 \left\|\overline{\varrho}_1\right\|_1 .
\end{aligned}$$

Furthermore, if $\overline{\lambda} = \lambda$, then it is clear that $\left\|\overline{\lambda}\overline{\varrho}_1\right\|_1 = \left|\overline{\lambda}\right|_1 \left\|\overline{\varrho}_1\right\|_1$.

(*iii*) From the definition of the relation $\leq_{\mathbb{D}}$, it is easy to see

$$\begin{aligned}\left\|\overline{\varrho}_1+\overline{\varrho}_2\right\|_1 &= \left\|\varrho_1+\varrho_2+\varepsilon\left(\varrho_1^*+\varrho_2^*\right)\right\|_1\\ &= \|\varrho_1+\varrho_2\|+\varepsilon\left\|\varrho_1^*+\varrho_2^*\right\|\\ &\leq_{\mathbb{D}} \|\varrho_1\|+\|\varrho_2\|+\varepsilon\left(\left\|\varrho_1^*\right\|+\left\|\varrho_2^*\right\|\right)\\ &= \left\|\overline{\varrho}_1\right\|_1+\left\|\overline{\varrho}_2\right\|_1.\end{aligned}$$

□

Definition 3.6. The function $\|\cdot\|_1 : \mathbb{D}^n \to \mathbb{D}$, $\left\|\overline{\varrho}_1\right\|_1 = \|\varrho_1\|+\varepsilon\left\|\varrho_1^*\right\|$ is called dual degenerate norm on $\mathbb{D}^n$, where the notation $\|\cdot\|$ is Euclidean norm on $\mathbb{R}^n$.

Remark 3.7. It is well known that $\left(\mathbb{D}^n,+,(\mathbb{D},+,\cdot),\cdot\right)$ is a module. If, instead of taking $(\mathbb{D},+,\cdot)$ as a commutative ring with unity, we consider the field of real numbers $(\mathbb{R},+,\cdot)$, then $\left(\mathbb{D}^n,+,(\mathbb{R},+,\cdot),\cdot\right)$ becomes a vector space. Consequently, for any $\overline{\varrho}_1,\overline{\varrho}_2\in\mathbb{D}^n$, the following properties hold:
(i) $\left\|\overline{\varrho}_1\right\|_1 \geq_{\mathbb{D}} \overline{0}$; equality holds if and only if $\overline{\varrho}_1=\overline{0}$.
(ii) For any $\lambda\in\mathbb{R}$, $\left\|\lambda\overline{\varrho}_1\right\|_1 = |\lambda|\left\|\overline{\varrho}_1\right\|_1$.
(iii) $\left\|\overline{\varrho}_1+\overline{\varrho}_2\right\|_1 \leq_{\mathbb{D}} \left\|\overline{\varrho}_1\right\|_1+\left\|\overline{\varrho}_2\right\|_1$.

Proposition 3.8. *Consider that the function $\overline{d}:\mathbb{D}^n\times\mathbb{D}^n\to\mathbb{D}$ is defined as $\overline{d}\left(\overline{\varrho}_1,\overline{\varrho}_2\right)=d_1\left(\varrho_1,\varrho_2\right)+\varepsilon d_2\left(\varrho_1^*,\varrho_2^*\right)$, where d_1 and d_2 are two metrics on $\mathbb{R}^n$. The function $\overline{d}$ holds the below properties.*
(i) $\overline{d}\left(\overline{\varrho}_1,\overline{\varrho}_2\right)\geq_{\mathbb{D}}\overline{0}$ for all $\overline{\varrho}_1,\overline{\varrho}_2\in\mathbb{D}^n$; equality holds if and only if $\overline{\varrho}_1=\overline{\varrho}_2$.
(ii) $\overline{d}\left(\overline{\varrho}_1,\overline{\varrho}_2\right)=\overline{d}\left(\overline{\varrho}_2,\overline{\varrho}_1\right)$ for all $\overline{\varrho}_1,\overline{\varrho}_2\in\mathbb{D}^n$.
(iii) (Triangle inequality) $\overline{d}\left(\overline{\varrho}_1,\overline{\varrho}_3\right)\leq_{\mathbb{D}}\overline{d}\left(\overline{\varrho}_1,\overline{\varrho}_2\right)+\overline{d}\left(\overline{\varrho}_2,\overline{\varrho}_3\right)$ for all $\overline{\varrho}_1,\overline{\varrho}_2,\overline{\varrho}_3\in\mathbb{D}^n$.

With these conventions, the function $\overline{d}:\mathbb{D}^n\times\mathbb{D}^n\to\mathbb{D}$, $\overline{d}\left(\overline{\varrho}_1,\overline{\varrho}_2\right)=d_1\left(\varrho_1,\varrho_2\right)+\varepsilon d_2\left(\varrho_1^*,\varrho_2^*\right)$ is called a dual-valued metric on $\mathbb{D}^n$ [3]. Considering the Proposition 3.8 and the relation $\leq_{\mathbb{D}}$, we have

Proposition 3.9 ([3]). *Let $\left(\mathbb{D}^n,\overline{d}\right)$ be a dual-valued metric space. Then, the collection of all the sets $\overline{\varphi}_1$ and $\overline{\varphi}_2$ is a basis for a topology on $\mathbb{D}^n$:*

$$\begin{aligned}\overline{\varphi}_1 &= \left\{\overline{\varrho}_1=\varrho_1+\varepsilon\varrho_1^*\in\mathbb{D}^n \mid d_1\left(\varrho_1,q\right)<r,\varrho_1^*\in\mathbb{R}^n\right\}\\ \overline{\varphi}_2 &= \left\{\overline{\varrho}_1=\varrho_1+\varepsilon\varrho_1^*\in\mathbb{D}^n \mid \varrho_1=c\,(constant), d_2\left(\varrho_1^*,q^*\right)<r^*\right\},\end{aligned}$$

where the metrics d_1 and d_2 are two metrics on $\mathbb{R}^n$, $\overline{q}=q+\varepsilon q^*\in\mathbb{D}^n$ and $\overline{r}=r+\varepsilon r^*\in\mathbb{D}^{+\prime}$.

By considering Propositions 3.8 and 3.9, we obtain the following result.

Corollary 3.10 ([3]). *The basis given in the Proposition 3.9 is symbolized as $\overline{\Phi}$. An arbitrary union of the sets in the basis $\overline{\Phi}$ is a topology denoted by $\overline{\tau}$, that is, $\overline{\tau}=\left\{\bigcup_{i\in I}\overline{\varphi}_i \mid \overline{\varphi}_i\in\overline{\Phi}\right\}$. Considering the set*

$$\begin{aligned}\overline{A} &= \left\{\overline{\varrho}_1=\varrho_1+\varepsilon\varrho_1^*\in\mathbb{D}^n \mid \overline{d}\left(\overline{\varrho}_1,\overline{q}\right)<_{\mathbb{D}}\overline{r}\right\}\\ &= \left\{\overline{\varrho}_1=\varrho_1+\varepsilon\varrho_1^*\in\mathbb{D}^n \mid d_1\left(\varrho_1,q\right)<r,\varrho_1^*\in\mathbb{R}^n\right\}\cup\left\{\overline{\varrho}_1=\varrho_1+\varepsilon\varrho_1^*\in\mathbb{D}^n \mid d_1\left(\varrho_1,q\right)=r,d_2\left(\varrho_1^*,q^*\right)<r^*\right\},\end{aligned}$$

it is clear that $\overline{A}\in\overline{\tau}$.

Remark 3.11 ([3]). The topological space $\left(\mathbb{D}^n,\overline{\tau}\right)$ is a Hausdorff space.

Now, let us create two relations $<_1$ and $\leq_1$ between dual numbers inspired by the article [2], and examine some of their fundamental properties. We will then give the definition of a different dual-valued metric using the relation $\leq_1$ and we will use this metric and the relation $<_1$ to obtain the topology denoted by $\overline{\tau}_1$ in the n-dimensional dual space $\mathbb{D}^n$ such that the space $\left(\mathbb{D}^n,\overline{\tau}_1\right)$ is shown to be a Hausdorff space. In addition, we shall compare the topologies $\overline{\tau}$ and $\overline{\tau}_1$, and demonstrate that $\overline{\tau}_1\subseteq\overline{\tau}$.

Definition 3.12. Consider that $\overline{\omega}=\omega+\varepsilon\omega^*$ and $\overline{\upsilon}=\upsilon+\varepsilon\upsilon^*$ are dual numbers. The relations $\overline{\omega}<_1\overline{\upsilon}$ and $\overline{\omega}\leq_1\overline{\upsilon}$ between these dual numbers are expressed as follows:
(*i*) $\overline{\omega}<_1\overline{\upsilon}\Leftrightarrow\omega<\upsilon$ and $\omega^*<\upsilon^*$.
(*ii*) $\overline{\omega}\leq_1\overline{\upsilon}\Leftrightarrow\omega\leq\upsilon$ and $\omega^*\leq\upsilon^*$.

Note that according to the relation $<_{\mathbb{D}}$, all dual numbers can be compared, while according to the relation $<_1$, this is not possible. However, we will see that the relation $<_1$ has an important place in the dual number system. By means of this relation, a new basis, and consequently a topology, will be obtained in the dual space, and the resulting data will enable us to make topological and geometrical interpretations of the space. At this point, we shall address propositions concerning certain fundamental properties associated with the relations $<_1$ and $\leq_1$.

Proposition 3.13. *Let $\overline{\omega} = \omega + \varepsilon\omega^*$, $\overline{\upsilon} = \upsilon + \varepsilon\upsilon^*$ and $\overline{\eta} = \eta + \varepsilon\eta^*$ be dual numbers. The relations $<_1$ and $\leq_1$ hold the below expressions.*

(i) If $\overline{\omega} <_1 \overline{\upsilon}$, then $\overline{\omega} \neq \overline{\upsilon}$.
(ii) If $\overline{\omega} <_1 \overline{\upsilon}$ and $\overline{\upsilon} <_1 \overline{\eta}$, then $\overline{\omega} <_1 \overline{\eta}$.
(iii) $\overline{\omega} \leq_1 \overline{\omega}$.
(iv) If $\overline{\omega} \leq_1 \overline{\upsilon}$ and $\overline{\upsilon} \leq_1 \overline{\omega}$, then $\overline{\omega} = \overline{\upsilon}$.
(v) If $\overline{\omega} \leq_1 \overline{\upsilon}$ and $\overline{\upsilon} \leq_1 \overline{\eta}$, then $\overline{\omega} \leq_1 \overline{\eta}$.

Proof. Using the Definition 3.12, the existence of each expression is attained. □

Proposition 3.14. *Suppose that $\overline{\omega}, \overline{\upsilon} \in \mathbb{D}$ and $\overline{\omega} <_1 \overline{\upsilon}$ (resp. $\overline{\omega} \leq_1 \overline{\upsilon}$). For $\overline{\eta} \in \mathbb{D}$, there exist the dual inequalities $\overline{\omega} \mp \overline{\eta} <_1 \overline{\upsilon} \mp \overline{\eta}$ (resp. $\overline{\omega} \mp \overline{\eta} \leq_1 \overline{\upsilon} \mp \overline{\eta}$).*

Proposition 3.15. *Let $\overline{\omega} = \omega + \varepsilon\omega^*$ and $\overline{\upsilon} = \upsilon + \varepsilon\upsilon^*$ be dual numbers. Then, the following expressions hold.*

(i) If $\overline{\omega} <_1 \overline{\upsilon}$ and $\overline{\eta} = \eta + \varepsilon\eta^* \in \mathbb{D}^{+\prime}$, then $\overline{\omega} \cdot \overline{\eta} <_1 \overline{\upsilon} \cdot \overline{\eta}$.
(ii) If $\overline{\omega} <_1 \overline{\upsilon}$ and $\overline{\eta} = \eta + \varepsilon\eta^* \in \mathbb{D}^{+\prime\prime}$, then $\dfrac{\overline{\omega}}{\overline{\eta}} <_1 \dfrac{\overline{\upsilon}}{\overline{\eta}}$.
(iii) If $\overline{\omega} \leq_1 \overline{\upsilon}$ and $\overline{\eta} = \eta + \varepsilon\eta^* \in \mathbb{D}^{+\prime}$, then $\overline{\omega} \cdot \overline{\eta} \leq_1 \overline{\upsilon} \cdot \overline{\eta}$.
(iv) If $\overline{\omega} \leq_1 \overline{\upsilon}$ and $\overline{\eta} = \eta + \varepsilon\eta^* \in \mathbb{D}^{+\prime\prime}$, then $\dfrac{\overline{\omega}}{\overline{\eta}} \leq_1 \dfrac{\overline{\upsilon}}{\overline{\eta}}$.

Proof. (*i*) Suppose that $\overline{\omega} <_1 \overline{\upsilon}$ and $\overline{\eta} = \eta + \varepsilon\eta^* \in \mathbb{D}^{+\prime}$. Since $\overline{\omega} <_1 \overline{\upsilon}$, we can write $\omega < \upsilon$ and $\omega^* < \upsilon^*$. Because of the fact that η and η^* are positive real numbers, we have $\omega\eta < \upsilon\eta$, $\omega\eta^* < \upsilon\eta^*$ and $\omega^*\eta < \upsilon^*\eta$. Then, it is clear that $\overline{\omega} \cdot \overline{\eta} <_1 \overline{\upsilon} \cdot \overline{\eta}$.

(*ii*) Assume that $\overline{\omega} <_1 \overline{\upsilon}$ and $\overline{\eta} = \eta + \varepsilon\eta^* \in \mathbb{D}^{+\prime\prime}$. From $\overline{\omega} <_1 \overline{\upsilon}$, the expressions $\omega < \upsilon$ and $\omega^* < \upsilon^*$ are obtained. Since η and η^* are positive and negative real numbers, respectively, there exist the relations $\dfrac{\omega}{\eta} < \dfrac{\upsilon}{\eta}$, $\dfrac{\omega^*}{\eta} < \dfrac{\upsilon^*}{\eta}$ and $\dfrac{\omega\eta^*}{\eta^2} > \dfrac{\upsilon\eta^*}{\eta^2}$. It is possible to say that $\dfrac{\omega}{\eta} < \dfrac{\upsilon}{\eta}$ and $\dfrac{\omega^*}{\eta} - \dfrac{\omega\eta^*}{\eta^2} < \dfrac{\upsilon^*}{\eta} - \dfrac{\upsilon\eta^*}{\eta^2}$. With these conventions, it is easy to see $\dfrac{\overline{\omega}}{\overline{\eta}} <_1 \dfrac{\overline{\upsilon}}{\overline{\eta}}$.

The proofs for the situations (*iii*) and (*iv*) are made as in (*i*) and (*ii*), respectively. □

Proposition 3.16. *Let $\overline{\omega} = \omega + \varepsilon\omega^*$ and $\overline{\upsilon} = \upsilon + \varepsilon\upsilon^*$ be dual numbers. Then, the following statements hold.*

(i) If $\overline{\omega} <_1 \overline{\upsilon}$ and $\overline{\eta} = \eta + \varepsilon\eta^* \in \mathbb{D}^{-\prime}$, then $\overline{\omega} \cdot \overline{\eta} >_1 \overline{\upsilon} \cdot \overline{\eta}$.
(ii) If $\overline{\omega} <_1 \overline{\upsilon}$ and $\overline{\eta} = \eta + \varepsilon\eta^* \in \mathbb{D}^{-\prime\prime}$, then $\dfrac{\overline{\omega}}{\overline{\eta}} >_1 \dfrac{\overline{\upsilon}}{\overline{\eta}}$.
(iii) If $\overline{\omega} \leq_1 \overline{\upsilon}$ and $\overline{\eta} = \eta + \varepsilon\eta^* \in \mathbb{D}^{-\prime}$, then $\overline{\omega} \cdot \overline{\eta} \geq_1 \overline{\upsilon} \cdot \overline{\eta}$.
(iv) If $\overline{\omega} \leq_1 \overline{\upsilon}$ and $\overline{\eta} = \eta + \varepsilon\eta^* \in \mathbb{D}^{-\prime\prime}$, then $\dfrac{\overline{\omega}}{\overline{\eta}} \geq_1 \dfrac{\overline{\upsilon}}{\overline{\eta}}$.

Proof. The proofs for these situations can be made in similar way to the Proposition 3.15. □

Proposition 3.17. *Let us assume $\overline{\omega} <_1 \overline{\upsilon}$ for all $\overline{\omega}, \overline{\upsilon} \in \mathbb{D}^{+\prime}$. For $1 \leq n \in \mathbb{N}$, the following statements hold.*

(i) $\overline{0} <_1 \overline{\omega}^{2n} <_1 \overline{\upsilon}^{2n}$.
(ii) $\overline{0} <_1 \overline{\omega}^{2n+1} <_1 \overline{\upsilon}^{2n+1}$.

Proof. (*i*) Assume that $\overline{0} <_1 \overline{\omega} <_1 \overline{\upsilon}$ for all $\overline{\omega}, \overline{\upsilon} \in \mathbb{D}^{+\prime}$. We know that $\omega^{2n} < \upsilon^{2n}$ and $2n\omega^{2n-1} < 2n\upsilon^{2n-1}$. It is obvious that $\omega^{2n} < \upsilon^{2n}$ and $2n\omega^*\omega^{2n-1} < 2n\upsilon^*\upsilon^{2n-1}$. Considering the relation $<_1$, this immediately implies that $\overline{0} <_1 \overline{\omega}^{2n} <_1 \overline{\upsilon}^{2n}$.

(*ii*) The proof for this situation is as in (*i*). □

Proposition 3.18. *Observe that $\overline{\omega} <_1 \overline{\upsilon}$ for all $\overline{\omega}, \overline{\upsilon} \in \mathbb{D}^{+\prime\prime}$. Then, there exists the inequality $\frac{1}{\overline{\omega}} >_1 \frac{1}{\overline{\upsilon}} >_1 \overline{0}$.*

Proof. From the hypothesis, we have $0 < \omega < \upsilon$ and $\omega^* < \upsilon^* < 0$. It is obvious that $0 < \frac{1}{\upsilon} < \frac{1}{\omega}$ and $-\frac{\omega^*}{\omega^2} > -\frac{\upsilon^*}{\upsilon^2} > 0$. Then, considering the relation $<_1$, we can write $\frac{1}{\overline{\omega}} >_1 \frac{1}{\overline{\upsilon}} >_1 \overline{0}$. □

Proposition 3.19. *Consider that $\overline{\omega} <_1 \overline{\upsilon}$ for all $\overline{\omega}, \overline{\upsilon} \in \mathbb{D}^{-\prime}$. For $1 \leq n \in \mathbb{N}$, there exist the following situations.*
(i) $\overline{\omega}^{2n} >_1 \overline{\upsilon}^{2n} >_1 \overline{0}$.
(ii) $\overline{\omega}^{2n+1} <_1 \overline{\upsilon}^{2n+1} <_1 \overline{0}$.

Proof. The proof is straightforward. □

Proposition 3.20. *Suppose that $\overline{\omega} <_1 \overline{\upsilon}$ for all $\overline{\omega}, \overline{\upsilon} \in \mathbb{D}^{-\prime\prime}$. Then, there is the dual inequality $\overline{0} >_1 \frac{1}{\overline{\omega}} >_1 \frac{1}{\overline{\upsilon}}$.*

Proof. Let us take $\overline{\omega} <_1 \overline{\upsilon}$ for all $\overline{\omega}, \overline{\upsilon} \in \mathbb{D}^{-\prime\prime}$. It is possible to say that $\omega < \upsilon < 0$ and $0 < \omega^* < \upsilon^*$. Considering the expressions $\frac{1}{\overline{\omega}} = \frac{1}{\omega} - \varepsilon\frac{\omega^*}{\omega^2}$ and $\frac{1}{\overline{\upsilon}} = \frac{1}{\upsilon} - \varepsilon\frac{\upsilon^*}{\upsilon^2}$ together with the hypothesis, $\frac{1}{\upsilon} < \frac{1}{\omega} < 0$ and $0 > -\frac{\omega^*}{\omega^2} > -\frac{\upsilon^*}{\upsilon^2}$ such that $\overline{0} >_1 \frac{1}{\overline{\omega}} >_1 \frac{1}{\overline{\upsilon}}$. □

Proposition 3.21. *Let $\overline{\omega} = \omega + \varepsilon\omega^*$ be a dual number, where $\omega^* \neq 0$. Then, $\overline{\omega} \in \overline{S} = S_1 \cup S_2 \Leftrightarrow \overline{\omega} >_1 \overline{\omega}^2$, where*

$$S_1 = \left\{ \overline{\omega} = \omega + \varepsilon\omega^* \in \mathbb{D} \,|0 < \omega < \frac{1}{2}, \omega^* \in \mathbb{R}^+ \right\},$$

$$S_2 = \left\{ \overline{\omega} = \omega + \varepsilon\omega^* \in \mathbb{D} \,|\frac{1}{2} < \omega < 1, \omega^* \in \mathbb{R}^- \right\}.$$

Proof. Let $\overline{\omega} = \omega + \varepsilon\omega^*$ and $\overline{\omega} \in \overline{S} = S_1 \cup S_2$. Then, there exist two situations:
(i) If $\overline{\omega} \in S_1$, since $0 < \omega < \frac{1}{2}$ and $\omega^* \in \mathbb{R}^+$, it is clear that $\overline{\omega} >_1 \overline{\omega}^2$.
(ii) If $\overline{\omega} \in S_2$, since $\frac{1}{2} < \omega < 1$ and $\omega^* \in \mathbb{R}^-$, it is clear that $\overline{\omega} >_1 \overline{\omega}^2$.
Conversely, suppose that $\overline{\omega} >_1 \overline{\omega}^2$. From the definition of the relation $<_1$, it is obvious that $\overline{\omega} \in \overline{S} = S_1 \cup S_2$. □

Corollary 3.22. *Let $\overline{\omega}$ and $\overline{\upsilon}$ be dual numbers. Then, using the definition of the relation $<_1$, there are the following expressions.*
(i) If $\overline{\omega}, \overline{\upsilon} \in \mathbb{D}^{+\prime}$, then $\overline{\omega} \cdot \overline{\upsilon} >_1 \overline{0}$.
(ii) If $\overline{\omega} \in \mathbb{D}^{+\prime}$ and $\overline{\upsilon} \in \mathbb{D}^{-\prime}$, then $\overline{\omega} \cdot \overline{\upsilon} <_1 \overline{0}$.
(iii) If $\overline{\omega} \in \mathbb{D}^{-\prime}$ and $\overline{\upsilon} \in \mathbb{D}^{+\prime}$, then $\overline{\omega} \cdot \overline{\upsilon} <_1 \overline{0}$.
(iv) If $\overline{\omega}, \overline{\upsilon} \in \mathbb{D}^{-\prime}$, then $\overline{\omega} \cdot \overline{\upsilon} >_1 \overline{0}$.

Proposition 3.23. *For $\overline{\omega}, \overline{\upsilon} \in \mathbb{D}$, the following inequalities hold.*
(i) $|\overline{\omega}.\overline{\upsilon}|_1 \leq_1 |\overline{\omega}|_1 |\overline{\upsilon}|_1$.
(ii) $\left|\frac{\overline{\omega}}{\overline{\upsilon}}\right|_1 \geq_1 \frac{|\overline{\omega}|_1}{|\overline{\upsilon}|_1}$ *for* $\upsilon \neq 0$.

Proof. Using the definition of the relation $\leq_1$, these inequalities can be found. □

Corollary 3.24. *For any dual number $\overline{\omega} = \omega + \varepsilon\omega^*$ and $\overline{\delta} = \delta + \varepsilon\delta^* \in \mathbb{D}^{+\prime}$, we have*

$$|\overline{\omega}|_1 <_1 \overline{\delta} \Leftrightarrow -\overline{\delta} <_1 \overline{\omega} <_1 \overline{\delta}.$$

Proposition 3.25. *Let $\{\overline{\omega}_1, \overline{\omega}_2, ..., \overline{\omega}_n\}$ and $\{\overline{\upsilon}_1, \overline{\upsilon}_2, ..., \overline{\upsilon}_n\}$ be any two sets of dual numbers where $\overline{\omega}_i = \omega_i + \varepsilon\omega_i^*$, $\overline{\upsilon}_i = \upsilon_i + \varepsilon\upsilon_i^*$ and $1 \leq i \leq n$, such that either $\overline{\omega}_1 \geq_1 \overline{\omega}_2 \geq_1 ... \geq_1 \overline{\omega}_n$ and $\overline{\upsilon}_1 \geq_1 \overline{\upsilon}_2 \geq_1 ... \geq_1 \overline{\upsilon}_n$ or $\overline{\omega}_1 \leq_1 \overline{\omega}_2 \leq_1 ... \leq_1 \overline{\omega}_n$ and $\overline{\upsilon}_1 \leq_1 \overline{\upsilon}_2 \leq_1 ... \leq_1 \overline{\upsilon}_n$, then*

$$\left(\frac{\overline{\omega}_1 + \overline{\omega}_2 + ... + \overline{\omega}_n}{n}\right)\left(\frac{\overline{\upsilon}_1 + \overline{\upsilon}_2 + ... + \overline{\upsilon}_n}{n}\right) \leq_1 \frac{1}{n}\sum_{k=1}^{n} \overline{\omega}_k\overline{\upsilon}_k.$$

Proof. The proof is straightforward. □

Proposition 3.26. *Let $\{\overline{\omega}_1, \overline{\omega}_2, ..., \overline{\omega}_n\}$ and $\{\overline{\upsilon}_1, \overline{\upsilon}_2, ..., \overline{\upsilon}_n\}$ be any two sets of dual numbers, where $\overline{\omega}_i = \omega_i + \varepsilon\omega_i^*$, $\overline{\upsilon}_i = \upsilon_i + \varepsilon\upsilon_i^*$ and $1 \le i \le n$. Then, there exists the following dual inequality*

$$\sum_{k=1}^{n} |\overline{\omega}_k + \overline{\upsilon}_k|_1 \le_1 \sum_{k=1}^{n} |\overline{\omega}_k|_1 + \sum_{k=1}^{n} |\overline{\upsilon}_k|_1 .$$

Proof. Using the Definition 3.1, we have

$$\begin{aligned}\sum_{k=1}^{n} |\overline{\omega}_k + \overline{\upsilon}_k|_1 &= |\overline{\omega}_1 + \overline{\upsilon}_1|_1 + |\overline{\omega}_2 + \overline{\upsilon}_2|_1 + ... + |\overline{\omega}_n + \overline{\upsilon}_n|_1 \\ &= |\omega_1 + \upsilon_1| + |\omega_2 + \upsilon_2| + ... + |\omega_n + \upsilon_n| \\ &+ \varepsilon\left(\left|\omega_1^* + \upsilon_1^*\right| + \left|\omega_2^* + \upsilon_2^*\right| + ... + \left|\omega_n^* + \upsilon_n^*\right|\right)\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{n} |\overline{\omega}_k|_1 + \sum_{k=1}^{n} |\overline{\upsilon}_k|_1 &= |\overline{\omega}_1|_1 + |\overline{\omega}_2|_1 + ... + |\overline{\omega}_n|_1 \\ &+ |\overline{\upsilon}_1|_1 + |\overline{\upsilon}_2|_1 + ... + |\overline{\upsilon}_n|_1 \\ &= |\omega_1| + |\omega_2| + ... + |\omega_n| + |\upsilon_1| + |\upsilon_2| + ... + |\upsilon_n| \\ &+ \varepsilon\left(\left|\omega_1^*\right| + \left|\omega_2^*\right| + ... + \left|\omega_n^*\right| + \left|\upsilon_1^*\right| + \left|\upsilon_2^*\right| + ... + \left|\upsilon_n^*\right|\right).\end{aligned}$$

Considering the fact that

$$|\omega_1 + \upsilon_1| + |\omega_2 + \upsilon_2| + ... + |\omega_n + \upsilon_n| \le |\omega_1| + |\omega_2| + ... + |\omega_n| + |\upsilon_1| + |\upsilon_2| + ... + |\upsilon_n|$$

and

$$\left|\omega_1^* + \upsilon_1^*\right| + \left|\omega_2^* + \upsilon_2^*\right| + ... + \left|\omega_n^* + \upsilon_n^*\right| \le \left|\omega_1^*\right| + \left|\omega_2^*\right| + ... + \left|\omega_n^*\right| + \left|\upsilon_1^*\right| + \left|\upsilon_2^*\right| + ... + \left|\upsilon_n^*\right|,$$

we obtain the inequality

$$\sum_{k=1}^{n} |\overline{\omega}_k + \overline{\upsilon}_k|_1 \le_1 \sum_{k=1}^{n} |\overline{\omega}_k|_1 + \sum_{k=1}^{n} |\overline{\upsilon}_k|_1 . \qquad \square$$

Proposition 3.27. *Let $\{\overline{\omega}_1, \overline{\omega}_2, ..., \overline{\omega}_n\}$ and $\{\overline{\upsilon}_1, \overline{\upsilon}_2, ..., \overline{\upsilon}_n\}$ be any two sets of dual numbers, where $\overline{\omega}_i = \omega_i + \varepsilon\omega_i^*$, $\overline{\upsilon}_i = \upsilon_i + \varepsilon\upsilon_i^*$ and $1 \le i \le n$. Assume that the sequences $\omega_1, \omega_2, ..., \omega_n$ and $\upsilon_1, \upsilon_2, ..., \upsilon_n$ are proportional. For $1 < p \in \mathbb{N}$, the following inequality holds*

$$\left(\sum_{k=1}^{n} |\overline{\omega}_k + \overline{\upsilon}_k|_1^p\right)^{\frac{1}{p}} \le_1 \left(\sum_{k=1}^{n} |\overline{\omega}_k|_1^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |\overline{\upsilon}_k|_1^p\right)^{\frac{1}{p}} .$$

Proof. Using the Definition 3.1, we find

$$\begin{aligned}\left(\sum_{k=1}^{n} |\overline{\omega}_k + \overline{\upsilon}_k|_1^p\right)^{\frac{1}{p}} &= \left(\sum_{k=1}^{n} \left(|\omega_k + \upsilon_k| + \varepsilon\left|\omega_k^* + \upsilon_k^*\right|\right)^p\right)^{\frac{1}{p}} \\ &= \left(|\omega_1 + \upsilon_1|^p + ... + |\omega_n + \upsilon_n|^p\right)^{\frac{1}{p}} \\ &+ \varepsilon\frac{\left|\omega_1^* + \upsilon_1^*\right||\omega_1 + \upsilon_1|^{p-1} + ... + \left|\omega_n^* + \upsilon_n^*\right||\omega_n + \upsilon_n|^{p-1}}{\left(|\omega_1 + \upsilon_1|^p + ... + |\omega_n + \upsilon_n|^p\right)^{\frac{p-1}{p}}},\end{aligned}$$

$$\begin{aligned}\left(\sum_{k=1}^{n} |\overline{\omega}_k|_1^p\right)^{\frac{1}{p}} &= \left(\sum_{k=1}^{n} \left(|\omega_k| + \varepsilon\left|\omega_k^*\right|\right)^p\right)^{\frac{1}{p}} \\ &= \left(|\omega_1|^p + ... + |\omega_n|^p\right)^{\frac{1}{p}} \\ &+ \varepsilon\frac{\left|\omega_1^*\right||\omega_1|^{p-1} + ... + \left|\omega_n^*\right||\omega_n|^{p-1}}{\left(|\omega_1|^p + ... + |\omega_n|^p\right)^{\frac{p-1}{p}}}\end{aligned}$$

and

$$\left(\sum_{k=1}^{n}|\overline{\upsilon}_k|_1^p\right)^{\frac{1}{p}} = \left(\sum_{k=1}^{n}\left(|\upsilon_k|+\varepsilon\left|\upsilon_k^*\right|\right)^p\right)^{\frac{1}{p}}$$
$$= \left(|\upsilon_1|^p+\ldots+|\upsilon_n|^p\right)^{\frac{1}{p}}$$
$$+\varepsilon\frac{\left|\upsilon_1^*\right||\upsilon_1|^{p-1}+\ldots+\left|\upsilon_n^*\right||\upsilon_n|^{p-1}}{\left(|\upsilon_1|^p+\ldots+|\upsilon_n|^p\right)^{\frac{p-1}{p}}}.$$

From the hypothesis, observe that $\omega_i = \lambda\upsilon_i$ for $1 \leq i \leq n$. With this information, we have

$$\left(\sum_{k=1}^{n}|\overline{\omega}_k+\overline{\upsilon}_k|_1^p\right)^{\frac{1}{p}} = |\lambda+1|\left(|\upsilon_1|^p+\ldots+|\upsilon_n|^p\right)^{\frac{1}{p}}$$
$$+\varepsilon\frac{\left|\omega_1^*+\upsilon_1^*\right||\upsilon_1|^{p-1}+\ldots+\left|\omega_n^*+\upsilon_n^*\right||\upsilon_n|^{p-1}}{\left(|\upsilon_1|^p+\ldots+|\upsilon_n|^p\right)^{\frac{p-1}{p}}}$$

and

$$\left(\sum_{k=1}^{n}|\overline{\omega}_k|_1^p\right)^{\frac{1}{p}} = |\lambda|\left(|\upsilon_1|^p+\ldots+|\upsilon_n|^p\right)^{\frac{1}{p}}$$
$$+\varepsilon\frac{\left|\omega_1^*\right||\upsilon_1|^{p-1}+\ldots+\left|\omega_n^*\right||\upsilon_n|^{p-1}}{\left(|\upsilon_1|^p+\ldots+|\upsilon_n|^p\right)^{\frac{p-1}{p}}}.$$

With these conventions, it is possible to write

$$\left(\sum_{k=1}^{n}|\overline{\omega}_k+\overline{\upsilon}_k|_1^p\right)^{\frac{1}{p}} \leq_1 \left(\sum_{k=1}^{n}|\overline{\omega}_k|_1^p\right)^{\frac{1}{p}}+\left(\sum_{k=1}^{n}|\overline{\upsilon}_k|_1^p\right)^{\frac{1}{p}}. \qquad \square$$

Starting from the relation $\leq_1$, we define a metric that gives rise to a basis and consequently a topology. We then proceed to compare this topology with the one described in Proposition 3.9, and subsequently illustrate the new topology by constructing geometric models in $\mathbb{R}^3$ for some of its elements.

Proposition 3.28. *Suppose that $\mathbb{D}^n$ is $n-$dimensional dual space and the notation $\|\cdot\|$ is Euclidean norm on $\mathbb{R}^n$. The dual function $\|\cdot\|:\mathbb{D}^n\to\mathbb{D}$ defined as $\left\|\overline{\varrho}_1\right\|_1 = \|\varrho_1\|+\varepsilon\left\|\varrho_1^*\right\|$, for any $\overline{\varrho}_1\in\mathbb{D}^n$, satisfies the following properties:*

(i) $\left\|\overline{\varrho}_1\right\|_1 \geq_1 \overline{0}$ for all $\overline{\varrho}_1\in\mathbb{D}^n$; equality holds if and only if $\overline{\varrho}_1 = \overline{0}$.

(ii) $\left\|\overline{\lambda}\overline{\varrho}_1\right\|_1 \leq_1 \left|\overline{\lambda}\right|_1\left\|\overline{\varrho}_1\right\|_1$ for all $\overline{\varrho}_1\in\mathbb{D}^n$ and $\overline{\lambda}\in\mathbb{D}$. Specially, in case of $\overline{\lambda}=\lambda\in\mathbb{R}$, then $\left\|\overline{\lambda}\overline{\varrho}_1\right\|_1 = \left|\overline{\lambda}\right|_1\left\|\overline{\varrho}_1\right\|_1$.

(iii) $\left\|\overline{\varrho}_1+\overline{\varrho}_2\right\|_1 \leq_1 \left\|\overline{\varrho}_1\right\|_1+\left\|\overline{\varrho}_2\right\|_1$ for all $\overline{\varrho}_1,\overline{\varrho}_2\in\mathbb{D}^n$.

Proof. The proof can be made by a similar way to the proof of the Proposition 3. $\square$

Proposition 3.29. *Let d_1, d_2 be two metrics on $\mathbb{R}^n$. The dual function $\overline{d}$ identified by $\overline{d}\left(\overline{\varrho}_1,\overline{\varrho}_2\right) = d_1\left(\varrho_1,\varrho_2\right)+\varepsilon d_2\left(\varrho_1^*,\varrho_2^*\right)$ holds the below properties:*

(i) $\overline{d}\left(\overline{\varrho}_1,\overline{\varrho}_2\right)\geq_1\overline{0}$ for all $\overline{\varrho}_1,\overline{\varrho}_2\in\mathbb{D}^n$; equality holds if and only if $\overline{\varrho}_1=\overline{\varrho}_2$.

(ii) $\overline{d}\left(\overline{\varrho}_1,\overline{\varrho}_2\right)=\overline{d}\left(\overline{\varrho}_2,\overline{\varrho}_1\right)$ for all $\overline{\varrho}_1,\overline{\varrho}_2\in\mathbb{D}^n$.

(iii) (Triangle inequality) $\overline{d}\left(\overline{\varrho}_1,\overline{\varrho}_3\right)\leq_1\overline{d}\left(\overline{\varrho}_1,\overline{\varrho}_2\right)+\overline{d}\left(\overline{\varrho}_2,\overline{\varrho}_3\right)$ for all $\overline{\varrho}_1,\overline{\varrho}_2,\overline{\varrho}_3\in\mathbb{D}^n$.

Proof. Considering the properties for the notation of metric on $\mathbb{R}^n$ and the relation $\leq_1$, the proof can be shown. $\square$

Remark 3.30. Assume that d_1 and d_2 are two metrics on $\mathbb{R}^n$ and the dual function $\overline{d}$ is defined by $\overline{d}\left(\overline{\varrho}_1,\overline{\varrho}_2\right) = d_1\left(\varrho_1,\varrho_2\right)+\varepsilon d_2\left(\varrho_1^*,\varrho_2^*\right)$ for all $\overline{\varrho}_1,\overline{\varrho}_2\in\mathbb{D}^n$. Both the relations $\leq_{\mathbb{D}}$ and $\leq_1$ satisfy the properties of (dual) positivity and (dual) triangle inequality (see Propositions 3.8 and 3.29). This would inevitably lead to the definition of two different dual-valued metrics.

Proposition 3.31. *Observe that d_1 and d_2 are two metrics on $\mathbb{R}^n$. A collection of all the sets*

$$\begin{aligned}\overline{\Psi} &= \left\{\overline{\varrho}_1 = \varrho_1 + \varepsilon\varrho_1^* \in \mathbb{D}^n \mid \overline{d}\left(\overline{\varrho}_1, \overline{a}\right) <_1 \overline{r}\right\} \\ &= \{\overline{\varrho}_1 = \varrho_1 + \varepsilon\varrho_1^* \in \mathbb{D}^n \mid d_1\left(\varrho_1, a\right) < r, d_2\left(\varrho_1^*, a^*\right) < r^*\}\end{aligned}$$

constitutes a basis $\overline{\beta}_1$ on $\mathbb{D}^n$, where $\overline{a} = a + \varepsilon a^* \in \mathbb{D}^n$ and $\overline{r} = r + \varepsilon r^* \in \mathbb{D}^{+\prime}$.

Proof. It is sufficient to show that the two conditions required for a basis are satisfied.

(*i*) It is obvious that $\bigcup_{\overline{B}\in\overline{\beta}_1} \overline{B} = \mathbb{D}^n$.

(*ii*) The set $\overline{\Psi}_1 \cap \overline{\Psi}_2$ is an arbitrary union of some sets belonging to the class $\overline{\beta}_1$ for all $\overline{\Psi}_1, \overline{\Psi}_2 \in \overline{\beta}_1$ except for $\overline{\Psi}_1 \cap \overline{\Psi}_2 = \varnothing$. Assume that $\overline{\varrho}_2 = \varrho_2 + \varepsilon\varrho_2^*$ belongs to $\overline{\Psi}_1 \cap \overline{\Psi}_2$, where

$$\overline{\Psi}_1 = \{\overline{\varrho}_1 = \varrho_1 + \varepsilon\varrho_1^* \in \mathbb{D}^n \mid d_1\left(\varrho_1, a_1\right) < r_1, d_2\left(\varrho_1^*, a_1^*\right) < r_1^*\}$$

and

$$\overline{\Psi}_2 = \{\overline{\varrho}_1 = \varrho_1 + \varepsilon\varrho_1^* \in \mathbb{D}^n \mid d_1\left(\varrho_1, a_2\right) < r_2, d_2\left(\varrho_1^*, a_2^*\right) < r_2^*\}.$$

We can find

$$\overline{\varrho}_2 \in \overline{B} = \{\overline{\varrho}_1 = \varrho_1 + \varepsilon\varrho_1^* \in \mathbb{D}^n \mid d_1\left(\varrho_1, \varrho_2\right) < r, d_2\left(\varrho_1^*, \varrho_2^*\right) < r^*\} \subseteq \overline{\Psi}_1 \cap \overline{\Psi}_2$$

such that the set $\overline{B}$ belongs to the class $\overline{\beta}_1$. With these conventions, the class $\overline{\beta}_1$ is a basis on $\mathbb{D}^n$. □

Corollary 3.32. *An arbitrary union of the sets in the basis $\overline{\beta}_1$ mentioned in the Proposition 3.31 forms a topology on $\mathbb{D}^n$. This topology is symbolized as $\overline{\tau}_1$. The topology $\overline{\tau}_1$ is mathematically expressed as follows:*

$$\overline{\tau}_1 = \left\{\bigcup_{i\in I} \overline{\Psi}_i \mid \overline{\Psi}_i \in \overline{\beta}_1\right\}.$$

Remark 3.33. Using the relations $<_{\mathbb{D}}$ and $<_1$, we show that two different topologies, denoted by $\overline{\tau}$ and $\overline{\tau}_1$, can be obtained in the n-dimensional dual space $\mathbb{D}^n$. Moreover, it is clear that

$$\overline{\tau}_1 \subseteq \overline{\tau}. \tag{3.1}$$

This result is quite interesting. Because, considering Propositions 3.8 and 3.29, no matter what metrics d_1 and d_2 are on $\mathbb{R}^n$, there is always a relation (3.1) between the topologies $\overline{\tau}$ and $\overline{\tau}_1$ obtained using relations $<_{\mathbb{D}}$ and $<_1$, respectively.

Remark 3.34. Görgülü et. al. introduced the notion of the absolute value of a dual number and defined it as

$$|\overline{w}| = |w| + |w^*|,$$

where $\overline{w} = w + \varepsilon w^* \in \mathbb{D}$ [17]. The topology arising from this definition coincides with the one obtained in the present work by employing the definition of $|\cdot|_1$ together with the relation $<_1$. In fact, both approaches lead to the very same topology, namely the standard topology of $\mathbb{R}^2$.

Proposition 3.35. *The topological space $(\mathbb{D}^n, \overline{\tau}_1)$ is a Hausdorff space.*

Proof. For all $\overline{q}_1, \overline{q}_2 \in \mathbb{D}^n$ satisfying the condition $\overline{q}_1 \neq \overline{q}_2$, there are two situations. Firstly, assume that $q_1 \neq q_2$. Then, we can find two elements

$$\overline{q}_1 \in \overline{\Psi}_1 = \{\overline{\varrho}_1 = \varrho_1 + \varepsilon\varrho_1^* \in \mathbb{D}^n \mid d_1\left(\varrho_1, q_1\right) < r_1, d_2\left(\varrho_1^*, q_1^*\right) < r_1^*\} \in \overline{\tau}_1$$

and

$$\overline{q}_2 \in \overline{\Psi}_2 = \{\overline{\varrho}_1 = \varrho_1 + \varepsilon\varrho_1^* \in \mathbb{D}^n \mid d_1\left(\varrho_1, q_2\right) < r_2, d_2\left(\varrho_1^*, q_2^*\right) < r_2^*\} \in \overline{\tau}_1$$

of the topological space $\overline{\tau}_1$ such that $\overline{\Psi}_1 \cap \overline{\Psi}_2 = \varnothing$. Now, suppose that $q_1 = q_2$ and $q_1^* \neq q_2^*$. Thus, we obtain

$$\overline{q}_1 \in \overline{\Psi}_1 = \{\overline{\varrho}_1 = \varrho_1 + \varepsilon\varrho_1^* \in \mathbb{D}^n \mid d_1\left(\varrho_1, q_1\right) < r_1, d_2\left(\varrho_1^*, q_1^*\right) < r_1^*\} \in \overline{\tau}_1$$

and

$$\overline{q}_2 \in \overline{\Psi}_2 = \{\overline{\varrho}_1 = \varrho_1 + \varepsilon\varrho_1^* \in \mathbb{D}^n \mid d_1\left(\varrho_1, q_2\right) < r_2, d_2\left(\varrho_1^*, q_2^*\right) < r_2^*\} \in \overline{\tau}_1$$

such that $\overline{\Psi}_1 \cap \overline{\Psi}_2 = \varnothing$. With these conventions, the space $(\mathbb{D}^n, \overline{\tau}_1)$ is a Hausdorff space. □

Now, we provide geometric modeling of the set $\overline{\Psi}$ given in the Proposition 3.31 in $\mathbb{R}^3$ considering some metrics d_1 and d_2 in Euclidean space.

Example 3.36. Assume that two metrics d_1 and d_2 on $\mathbb{R}^2$ are defined by

$$\begin{array}{rcl} d_1 & : & \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \\ (x,y) & \to & d_1(x,y) = |x_1 - y_1| + |x_2 - y_2| \end{array}$$

and

$$\begin{array}{rcl} d_2 & : & \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \\ (x^*,y^*) & \to & d_2(x^*,y^*) = \|x^* - y^*\|, \end{array}$$

respectively. For any arbitrary $\overline{a} = a + \varepsilon a^* \in \mathbb{D}^2$ and $\overline{r} = r + \varepsilon r^* \in \mathbb{D}^{+\prime}$, the set $\overline{\Psi}$ is constituted as below:

$$\begin{aligned} \overline{\Psi} &= \left\{ \overline{x} = x + \varepsilon x^* \in \mathbb{D}^2 \mid \overline{d}(\overline{x},\overline{a}) <_1 \overline{r} \right\} \\ &= \left\{ \overline{x} = x + \varepsilon x^* \in \mathbb{D}^2 \mid |x_1 - a_1| + |x_2 - a_2| < r, \|x^* - a^*\| < r^* \right\}. \end{aligned}$$

In case of $\overline{a} = 0 + \varepsilon 0 \in \mathbb{D}^2$, the set $\overline{\Psi}$ is as follows:

$$\overline{\Psi} = \left\{ \overline{x} = x + \varepsilon x^* \in \mathbb{D}^2 \mid |x_1| + |x_2| < r, \|x^*\| < r^* \right\}.$$

Now, suppose that $\overline{x} = x + \varepsilon x^* \in \mathbb{D}^2$, where $x = (x_1, x_2)$ and $x^* = \left(x_1^*, 0\right)$. The following geometric modeling of the set $\overline{\Psi}$ is obtained (see Figure 1):

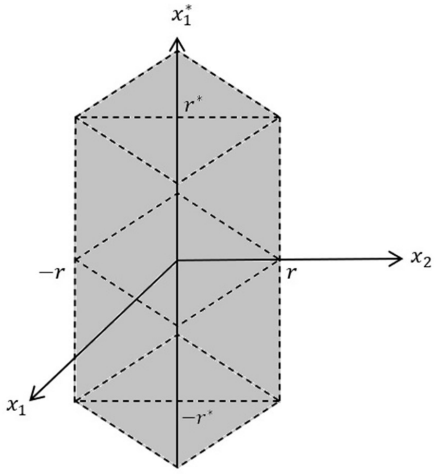


Figure 1. Basic open sets for the absolute value and Euclidean metrics.

Example 3.37. Consider that the metrics d_1 and d_2 on $\mathbb{R}^2$ are identified by

$$\begin{array}{rcl} d_1 & : & \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \\ (x,y) & \to & d_1(x,y) = |x_1 - y_1| + |x_2 - y_2| \end{array}$$

and

$$\begin{array}{rcl} d_2 & : & \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \\ (x^*,y^*) & \to & d_2(x^*,y^*) = \left\{ \begin{array}{ll} 0, & x^* = y^* \\ 1, & x^* \neq y^* \end{array} \right. , \end{array}$$

respectively. With respect to the situations of r^*, there are two cases for the set $\overline{\Psi}$.

- For $0 < r^* \leq 1$, we have

$$\begin{aligned}\overline{\Psi} &= \left\{\overline{x} = x + \varepsilon x^* \in \mathbb{D}^2 \mid \overline{d}(\overline{x}, \overline{a}) <_1 \overline{r}\right\} \\ &= \left\{\overline{x} = x + \varepsilon x^* \in \mathbb{D}^2 \mid |x_1 - a_1| + |x_2 - a_2| < r, x^* = a^*\right\}.\end{aligned}$$

- For $r^* > 1$, we have

$$\begin{aligned}\overline{\Psi} &= \left\{\overline{x} = x + \varepsilon x^* \in \mathbb{D}^2 \mid \overline{d}(\overline{x}, \overline{a}) <_1 \overline{r}\right\} \\ &= \left\{\overline{x} = x + \varepsilon x^* \in \mathbb{D}^2 \mid |x_1 - a_1| + |x_2 - a_2| < r, x^* \in \mathbb{R}^2\right\}.\end{aligned}$$

- For $0 < r^* \leq 1$ and $\overline{a} = 0 + \varepsilon 0 \in \mathbb{D}^2$, the set $\overline{\Psi}$ is as below:

$$\overline{\Psi} = \left\{\overline{x} = x + \varepsilon x^* \in \mathbb{D}^2 \mid |x_1| + |x_2| < r, x^* = 0\right\}.$$

- For $r^* > 1$ and $\overline{a} = 0 + \varepsilon 0 \in \mathbb{D}^2$, the set $\overline{\Psi}$ is generated by

$$\overline{\Psi} = \left\{\overline{x} = x + \varepsilon x^* \in \mathbb{D}^2 \mid |x_1| + |x_2| < r, x^* \in \mathbb{R}^2\right\}.$$

We constitute the geometric modeling of the set $\overline{\Psi}$ in $\mathbb{R}^3$ for the situations of r^*. For this, we observe $x = (x_1, x_2)$ and $x^* = \left(x_1^*, 0\right)$. Then, the following geometric modellings are obtained (see Figure 2).

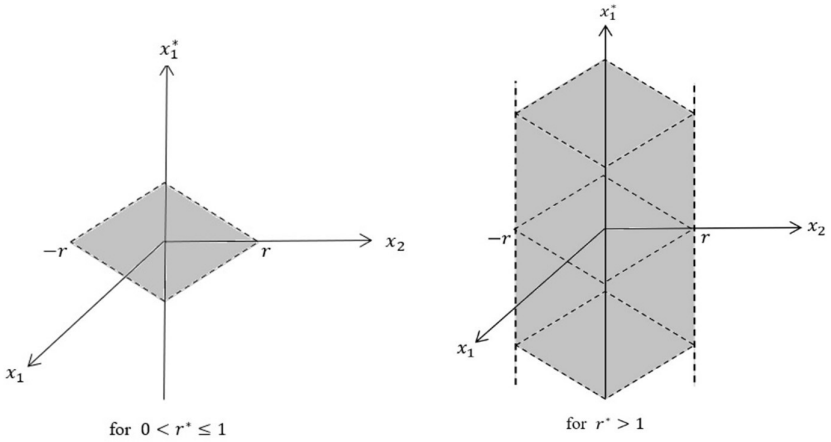


Figure 2. Basic open sets for the absolute value and discrete metrics.

4. Conclusion

In this paper, the dual number system is reclassified, and subsequently, a new system of inequalities is introduced. Fundamental properties of this system are examined, and the existence of certain specific inequalities (Minkowski, Chebyshev) is investigated within this space. Also, we have established two distinct topologies arising from different order relations on the set of dual numbers and demonstrated that both are Hausdorff. Furthermore, by comparing these topologies, it has been shown that $\overline{\tau}_1 \subseteq \overline{\tau}$. These findings provide a new perspective in the study of dual number systems and open possible directions for further investigations, particularly in the context of geometric modeling and applications in functional analysis.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The authors contributed to all aspects of the study, including the conception of the research, development of the theory, analysis of the results, and preparation of the manuscript.

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