



Equitable and Outdegree Equitable Domination in Product Graphs

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ABSTRACT. An equitable dominating set of a graph $G = (V, E)$, is a subset of vertices $D \subseteq V$ such that each vertex $v \in V - D$, is adjacent to at least one vertex $u \in D$ and the degree of u and v differ by at most 1. The equitable domination number $\gamma^e(G)$ is the size of the smallest equitable dominating set of G . For a vertex u in the dominating set D of a graph G , the number of edges from u to $V - D$ is called the outdegree of u with respect to D , $d_{D,G}^o(u)$. A dominating set D is called the outdegree equitable dominating set if the absolute value of the differences of outdegrees of any two vertices in D is at most one. The outdegree equitable domination number of G , $\gamma_{oe}(G)$ is the size of the smallest outdegree equitable dominating set of G . In this paper equitability related domination parameters such as equitable domination number and outdegree equitable domination number are analyzed within the realm of some graph products specifically the cartesian and the strong product of graphs.

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Keywords: Equitable dominating set, outdegree equitable dominating set, equitable domination number, outdegree equitable domination number.

1. INTRODUCTION

The concept of domination, introduced by Ore and Berge [2, 12], sparked a revolutionary movement in graph theory research, leading to significant advancements in the field, [1, 5, 9]. Resultantly different variations of domination were pioneered which entails total domination, independent domination, perfect domination, roman domination, italian domination, secure domination and so forth. The concept got wide acceptance due to its numerous applications in different fields comprising network security and monitoring, social network analysis, biology and epidemiology, transportation and logistics and more.

This paper exclusively considers finite, undirected, simple, and connected graphs. We adhere to the definitions and notations established in [4]. In a graph G , the neighborhood of a vertex v , denoted by $N(v)$ is the set of all vertices adjacent to v . The degree of v , represented as $d_G(v)$ is the number of edges incident to it.

A dominating set of a graph G is a subset D of vertices of G such that for each vertex u in $V - D$ there exists some vertex v in D such that $N(v)$ contains u . The domination number $\gamma(G)$ is the size of the smallest dominating set of G . A dominating set of G with size $\gamma(G)$ is termed as a γ - set of G .

An equitable dominating set of a graph $G = (V, E)$, is a subset of vertices $D \subseteq V$ such that each vertex $v \in V - D$, is adjacent to at least one vertex $u \in D$ and the degree of u and v differ by at most 1. We say that u equitably dominates

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v. The equitable domination number $\gamma^e(G)$ is the size of the smallest equitable dominating set of G . A γ^e -set of G is an equitable dominating set of G with exactly $\gamma^e(G)$ vertices.

A vertex $u \in V$ is an equitable isolate of G , if $|d_G(u) - d_G(v)| \geq 2$ for all $v \in N(u)$. An equitable isolate of a graph cannot be dominated equitably by any other vertex of the graph. Hence, every equitable dominating set of a graph contains the equitable isolates of the graph (if it exists).

Let D be a dominating set of a graph G . For a vertex $u \in D$, the number of edges with one end vertex u and the other end vertex $\in V - D$ is called the outdegree of u with respect to D , $d_{D,G}^o(u)$. A dominating set D is called the outdegree equitable dominating set if the absolute value of the differences of outdegrees of any two vertices of D is at most one. The outdegree equitable domination number of G $\gamma_{oe}(G)$ is the size of the smallest outdegree equitable dominating set of G . An outdegree equitable dominating set of G with exactly $\gamma_{oe}(G)$ vertices is termed as a γ_{oe} -set of G .

Example 1.1. In Figure 1,

$\gamma(G) = 2$, $\{v_2, v_3\}$ is a γ -set of G

$\gamma^e(G) = 3$, $\{v_3, v_5, v_6\}$ is a γ^e -set of G . Here v_3 and v_6 are equitable isolates.

$\gamma_{oe}(G) = 2$, $\{v_2, v_4\}$ is a γ_{oe} -set of G

$\{v_2, v_3\}$ is a dominating set of G but it is neither an equitable dominating set nor an outdegree equitable dominating set of G .

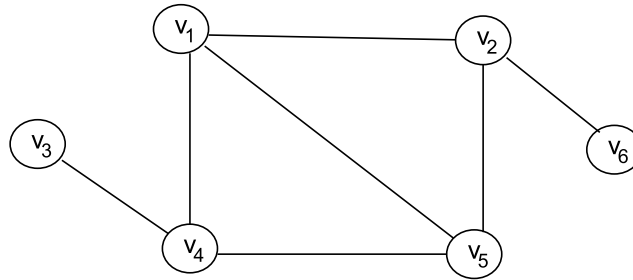


FIGURE 1. G

The concept of equitable domination was brought to light by V. Swaminathan and K.M. Dharmalingham, [14] whereas the idea of outdegree equitable domination was uncovered by K. M. Dharmalingham, [6]. Additional research on these topics can be discovered in the manuscripts [3, 10, 11, 13, 15–18].

The Cartesian product of two graphs G and H is the graph, denoted as $G \square H$, with vertex set $V(G) \times V(H)$ in which two vertices (g, h) and (g', h') are adjacent if and only if $g = g'$ and $hh' \in E(H)$ or $h = h'$ and $gg' \in E(G)$, [8]. A G -layer in $G \square H$, is the subgraph of $G \square H$ induced by the set of vertices $\{(g, h) : g \in V(G)\}$ where h is a fixed vertex from H . An H -layer in $G \square H$, is the subgraph of $G \square H$ induced by the set of vertices $\{(g, h) : h \in V(H)\}$ where g is a fixed vertex from G . Each G -layer is isomorphic to the graph G , whereas each H -layer is isomorphic to the graph H .

The strong product of two graphs G and H is the graph, denoted as $G \boxtimes H$, with vertex set $V(G) \times V(H)$ in which two vertices (g, h) and (g', h') are adjacent if and only if any one of the following condition holds.

1. $g = g'$ and $hh' \in E(H)$
2. $h = h'$ and $gg' \in E(G)$
3. $gg' \in E(G)$ and $hh' \in E(H)$



In this paper equitable domination and outdegree equitable domination is studied in the context of the cartesian and strong product of graphs. Established the upper bound for both domination parameters, characterized the cases for which equitable [outdegree equitable] domination number of cartesian [strong] product of graphs equals 1, 2. Also studied the concept in the case of regular graphs.

Remark 1.2. A universal vertex of a graph is a vertex that is adjacent to all other vertices of the graph. Trivially $\gamma_{oe}(G) = 1$ if and only if G has a universal vertex.

2. OUTDEGREE EQUITABLE DOMINATION NUMBER

Theorem 2.1. $\gamma_{oe}(G \square H) \leq \min\{|V(H)|\gamma_{oe}(G), |V(G)|\gamma_{oe}(H)\}$.

Proof. Let $V(G) = \{u_1, u_2, u_3, \dots, u_m\}$ and $V(H) = \{v_1, v_2, v_3, \dots, v_n\}$. Let D be a γ_{oe} -set of G . Let G_i denote the G -layer in $G \square H$, with the v_i chosen as the fixed vertex from H , $i = 1, 2, \dots, n$. Let $D_i = D \times v_i$. Then, D_i is a γ_{oe} -set of G_i . Suppose that $D' = \bigcup_{i=1}^n D_i$. Then $|D'| = \sum_{i=1}^n |D_i| = n|D| = |V(H)|\gamma_{oe}(G)$.

Claim : D' is an outdegree equitable dominating set of $G \square H$.

Clearly D' is a dominating set of $G \square H$, as for each $i = 1, 2, \dots, n$, D_i is a dominating set of G_i . It remains to prove that the outdegrees of vertices in D' are equitable. Let (u_p, v_q) and (u_s, v_t) are two elements in D' . Hence, $u_p, u_s \in D$ and as D being a γ_{oe} -set of G , we get

$$|d_{D,G}^o(u_p) - d_{D,G}^o(u_s)| \leq 1. \quad (2.1)$$

As $D' = D \times V(H)$, there are no outgoing edges from D' to $V(G \square H) - D'$ along the H components. Hence, $d_{D',G \square H}^o(u_p, v_p) = d_{D,G}^o(u_p)$ and $d_{D',G \square H}^o(u_s, v_t) = d_{D,G}^o(u_s)$. Therefore, from (2.1) we get

$$|d_{D',G \square H}^o(u_p, v_p) - d_{D',G \square H}^o(u_s, v_t)| = |d_{D,G}^o(u_p) - d_{D,G}^o(u_s)| \leq 1.$$

Thus, D' is an outdegree equitable dominating set of $G \square H$. Our claim is established. Therefore,

$$\gamma_{oe}(G \square H) \leq |V(H)|\gamma_{oe}(G). \quad (2.2)$$

Since cartesian product of graphs is commutative we get

$$\gamma_{oe}(H \square G) \leq |V(G)|\gamma_{oe}(H). \quad (2.3)$$

From (2.2) and (2.3),

$$\gamma_{oe}(G \square H) \leq \min\{|V(H)|\gamma_{oe}(G), |V(G)|\gamma_{oe}(H)\}. \quad \square$$

Corollary 2.2. $\gamma_{oe}(G \square H) = 1$ if and only if either $\gamma_{oe}(G) = 1$ and $H = K_1$ or $\gamma_{oe}(H) = 1$ and $G = K_1$.

Proof. If either $\gamma_{oe}(G) = 1$ and $H = K_1$ or $\gamma_{oe}(H) = 1$ and $G = K_1$, then from Theorem 2.1 we get that $\gamma_{oe}(G \square H) = 1$. Suppose that $\gamma_{oe}(G \square H) = 1$. This implies that $G \square H$ contains a universal vertex. This is possible if and only if either $G = K_1$ or $H = K_1$. Let $G = K_1$, then $G \square H$ is identical to H itself. Hence, $\gamma_{oe}(G \square H) = 1$ implies $\gamma_{oe}(H) = 1$. Suppose that $H = K_1$, then $G \square H$ is identical to G itself. Hence, $\gamma_{oe}(G \square H) = 1$ implies $\gamma_{oe}(G) = 1$. $\square$

Illustration 2.3. The upper bound established in the Theorem 2.1 is sharp. Figure 2 illustrates this.

Here $\gamma_{oe}(P_3) = \gamma_{oe}(P_2) = 1$. Darkened vertices forms a γ_{oe} -set of $P_3 \square P_2$ giving $\gamma_{oe}(P_3 \square P_2) = 2$.

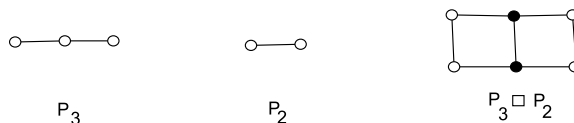


FIGURE 2.

Theorem 2.4. Let G be a graph with $|V(G)| \geq 3$, then $\gamma_{oe}(G \square P_n) = n$ if and only if $\gamma_{oe}(G) = 1$.

Proof. Let $P_n = v_1, v_2, \dots, v_n$. Let G_i denote the i -th G -layer in $G \square P_n$, with the v_i chosen as the fixed vertex from P_n , $i = 1, 2, \dots, n$. As $\gamma_{oe}(P_n) = \lceil \frac{n}{3} \rceil$, [6] from Theorem 2.1, we get that

$$\gamma_{oe}(G \square P_n) \leq \min \{n\gamma_{oe}(G), |V(G)| \lceil \frac{n}{3} \rceil\}. \quad (2.4)$$

Suppose that $\gamma_{oe}(G) = 1$, then (2.4) gives

$$\gamma_{oe}(G \square P_n) \leq \min \{n, |V(G)| \lceil \frac{n}{3} \rceil\}. \quad (2.5)$$

Since $|V(G)| \geq 3$, we can say that $n \leq |V(G)| \lceil \frac{n}{3} \rceil$. Therefore, from (2.5),

$$\gamma_{oe}(G \square P_n) \leq n. \quad (2.6)$$

If possible suppose that $\gamma_{oe}(G \square P_n) < n$. Let D be a γ_{oe} -set of $G \square P_n$. Let $D_i = V(G_i) \cap D$, then for $i \neq j$, $D_i \cap D_j = \emptyset$ and $D = \bigcup_{i=1}^n D_i$. Then, $|D| = \sum_{i=1}^n |D_i|$. By our assumption $|D| < n$. Therefore, $\sum_{i=1}^n |D_i| < n$ which implies that there exists at least one i such that $|D_i| = 0$. In this case in order to dominate the vertices of G_i , D should contain either all vertices of $G_{i-1} \cup G_{i+1}$ or at least $|V(G)|$ vertices from $V(G_{i-1}) \cup V(G_{i+1})$. This then gives that $|D| > n$, which is a contradiction to our assumption. Hence, from (2.6), we get $\gamma_{oe}(G \square P_n) = n$.

Conversely suppose that $\gamma_{oe}(G \square P_n) = n$ and let D denote a γ_{oe} -set of $G \square P_n$. By our assumption $|D| = n$. Let $D_i = V(G_i) \cap D$, then for $i \neq j$, $D_i \cap D_j = \emptyset$ and $D = \bigcup_{i=1}^n D_i$. Then, $|D| = \sum_{i=1}^n |D_i|$. Hence $|D| = n$ implies $\sum_{i=1}^n |D_i| = n$. If $|D_i| > 1$, for all $i = 1, 2, \dots, n$, then $|D| = \sum_{i=1}^n |D_i| > n$, which is a contradiction. Suppose that there exists some i , $i = 1, 2, \dots, n$ such that $|D_i| = 0$. In this case in order to dominate the vertices of G_i , D should contain either all vertices of $G_{i-1} \cup G_{i+1}$ or at least $|V(G)|$ vertices from $V(G_{i-1}) \cup V(G_{i+1})$. As $|V(G)| > 3$, this then gives that $|D| > n$, which is a contradiction to our assumption. Hence, $|D_i| = 1$, for all $i = 1, 2, \dots, n$. Suppose that the vertex $(u_i, v_i) \in D_i$ for all $i = 1, 2, \dots, n$. Then, (u_i, v_i) dominates all vertices of G_i for some i , otherwise $|D_i| > 1$ for at least one i , which is a contradiction. Hence, $\gamma(G_i) = 1$, which implies that $\gamma(G) = 1$. Hence, $\gamma_{oe}(G) = 1$. $\square$

Theorem 2.5. Let G and H be two graphs with at least two vertices, then $\gamma_{oe}(G \square H) = 2$ if and only if any one of the following condition holds.

- (a) $\gamma_{oe}(G) = 1$ and $H = K_2$.
- (b) $H = K_2$ and there exists two non adjacent vertices u, v in G such that $d_G(u) = d_G(v) = |V(G)| - 2$.

Proof. As G and H are graphs with at least two vertices, $G \square H$ contains no universal vertex. Hence $\gamma_{oe}(G \square H) \geq 2$. Suppose that $\gamma_{oe}(G) = 1$ and $H = K_2$. Then, by Theorem 2.1, we get $\gamma_{oe}(G \square H) \leq 2$. Therefore, $\gamma_{oe}(G \square H) = 2$.

Suppose that $H = K_2$ and there exists two non adjacent vertices u, v in G such that $d_G(u) = d_G(v) = |V(G)| - 2$. Let $V(H) = \{x, y\}$.

Claim : $D = \{(u, x), (v, y)\}$ is a γ_{oe} -set of $G \square H$.

Clearly D is a dominating set of $G \square H$. $d_{D, G \square H}^o(u, x) = |V(G)| - 2 + 1 = |V(G)| - 1$. Similarly $d_{D, G \square H}^o(v, y) = |V(G)| - 2 + 1 = |V(G)| - 1$. Therefore, $|d_{D, G \square H}^o(u, x) - d_{D, G \square H}^o(v, y)| \leq 1$. Thus, D is an outdegree equitable dominating set of $G \square H$ giving $\gamma_{oe}(G \square H) = 2$.

Conversely suppose that $\gamma_{oe}(G \square H) = 2$.

From [7],

$$\gamma(G \square H) \geq \min\{|V(G)|, |V(H)|\}.$$

Also,

$$\gamma(G \square H) \leq \gamma_{oe}(G \square H).$$

Therefore, from our assumption $\gamma_{oe}(G \square H) = 2$, we can say that at least one of the graphs has exactly 2 vertices. Suppose that H have two vertices which implies $H = K_2$. Let G_1, G_2 denote the G -layer of $G \square H$, with the fixed vertex of H chosen as u_1, u_2 respectively. Let D be a γ_{oe} -set of $G \square H$. By our assumption $|D| = 2$. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ and $V(H) = \{u_1, u_2\}$. Then, D is in one of the following form.

(1) $D = \{(v_j, u_1), (v_k, u_1)\}, j \neq k$.

(2) $D = \{(v_j, u_1), (v_k, u_2)\}, j \neq k$.

(3) $D = \{(v_j, u_1), (v_j, u_2)\}$.

Case 1. $D = \{(v_j, u_1), (v_k, u_1)\}, j \neq k$.

If possible, suppose that $|V(G)| \geq 3$. Then, G contains at least one vertex v_s other than v_j and v_k which implies that $G \square H$ contains at least two vertices $(v_s, u_1), (v_s, u_2)$. But then the vertex (v_s, u_2) is not dominated by D , which is a contradiction to the assumption that D is a γ_{oe} -set of $G \square H$. Hence, $|V(G)| = 2$. Thus, $G = K_2$ and $\gamma_{oe}(G) = 1$.

Case 2. $D = \{(v_j, u_1), (v_k, u_2)\}, j \neq k$.

Note that (v_k, u_2) can only dominate its own copy (v_k, u_1) in G_1 . Hence, (v_j, u_1) must dominate all the remaining vertices in G_1 . If (v_k, u_1) is adjacent to (v_j, u_1) , then v_j is a universal vertex in G and hence $\gamma_{oe}(G) = 1$.

Suppose that (v_k, u_1) is not adjacent to (v_j, u_1) . This implies that v_k is not adjacent to v_j in G . As v_j is adjacent to all the remaining vertices of G , we get $d_G(v_j) = |V(G)| - 2$. Similarly we can say that D being a dominating set of $G \square H$, all the vertices in G_2 except possibly (v_j, u_2) is adjacent to (v_k, u_2) . If (v_k, u_2) is adjacent to (v_j, u_2) , then v_k is a universal vertex in G and hence $\gamma_{oe}(G) = 1$. Suppose (v_k, u_2) is not adjacent to (v_j, u_2) . This implies that v_k is not adjacent to v_j in G . As v_k is adjacent to all the remaining vertices of G , we get $d_G(v_k) = |V(G)| - 2$. Thus, there exists two non adjacent vertices u, v in G such that

$$d_G(u) = d_G(v) = |V(G)| - 2.$$

Case 3. $D = \{(v_j, u_1), (v_j, u_2)\}$.

D being a dominating set of $G \square H$, the vertex (v_j, u_1) is adjacent to all the vertices in G_1 which implies that v_j is a universal vertex of G . Hence, $\gamma_{oe}(G) = 1$. $\square$

Illustration 2.6. Figure 3 illustrates case (b) in Theorem 2.4. Here, u and v are two non adjacent vertices with $d_{C_4}(u) = d_{C_4}(v) = 2 = |V(C_4)| - 2$. The darkened vertices in $C_4 \square K_2$ forms a γ_{oe} -set of $C_4 \square K_2$. Hence, $\gamma_{oe}(C_4 \square K_2) = 2$.

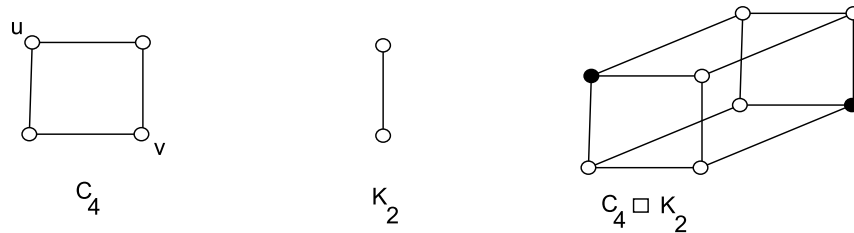


FIGURE 3.

Theorem 2.7. Let G be a k regular graph and H be an l regular graph. Suppose that there exists γ_{oe} sets D_1, D_2 of G, H , respectively, such that $d^o_{D_1}(v) = s$, for all $v \in D_1$ and $d^o_{D_2}(v) = t$, for all $v \in D_2$, then $\gamma_{oe}(G \boxtimes H) \leq \gamma_{oe}(G)\gamma_{oe}(H)$.

Proof. Let $D = D_1 \times D_2$. Then, $|D| = |D_1||D_2| = \gamma_{oe}(G)\gamma_{oe}(H)$. If $|D_1| = |D_2| = 1$, then both G and H has universal vertex and hence $G \boxtimes H$ has universal vertex. Therefore, $\gamma_{oe}(G) = \gamma_{oe}(H) = \gamma_{oe}(G \boxtimes H) = 1$ which gives $\gamma_{oe}(G \boxtimes H) = \gamma_{oe}(G)\gamma_{oe}(H)$. Suppose at least one of the sets D_1, D_2 has more than one vertex.

Claim : D is an outdegree equitable dominating set of $G \boxtimes H$.

Clearly D is a dominating set of $G \boxtimes H$. Let $(u, v), (x, y) \in D$.

$$d^o_D(u, v) = d^o_{D_1}(u) + d^o_{D_2}(v) + d_G(u)d^o_{D_2}(v) + d_H(v)d^o_{D_1}(u) - d^o_{D_1}(u)d^o_{D_2}(v). \quad (2.7)$$

Similarly

$$d^o_D(x, y) = d^o_{D_1}(x) + d^o_{D_2}(y) + d_G(x)d^o_{D_2}(y) + d_H(y)d^o_{D_1}(x) - d^o_{D_1}(x)d^o_{D_2}(y). \quad (2.8)$$

Now, $d_G(u) = d_G(x) = k$, as G is a k regular graph and $d_H(v) = d_H(y) = l$, as H is an l regular graph. As $d^o_{D_1}(u) = d^o_{D_1}(x) = s$ and $d^o_{D_2}(v) = d^o_{D_2}(y) = t$, from (2.7) and (2.8), we get

$$|d^o_D(u, v) - d^o_D(x, y)| = 0.$$

Hence, D is an outdegree equitable dominating set of $G \boxtimes H$ giving

$$\gamma_{oe}(G \boxtimes H) \leq |D| = \gamma_{oe}(G)\gamma_{oe}(H). \quad \square$$

3. EQUITABLE DOMINATION NUMBER

Theorem 3.1. $\gamma^e(G \square H) \leq \min\{|V(H)|\gamma^e(G), |V(G)|\gamma^e(H)\}.$

Proof. Let $V(G) = \{u_1, u_2, u_3, \dots, u_m\}$ and $V(H) = \{v_1, v_2, v_3, \dots, v_n\}$. Let D be a γ^e -set of G . Let G_i denote the i -th G -layer in $G \square H$, with the v_i chosen as the fixed vertex from H , $i = 1, 2, \dots, n$. Let $D_i = D \times v_i$. Then, D_i is a γ^e -set of G_i . Let $D' = \bigcup_{i=1}^n D_i$. Then, $|D'| = \sum_{i=1}^n |D_i| = n|D| = |V(H)|\gamma^e(G)$.

Claim : D' is an equitable dominating set of $G \square H$.

Clearly D' is a dominating set of $G \square H$, as for each $i = 1, 2, \dots, n$, D_i is a dominating set of G_i . Let $(u_t, v_k) \in V(G \square H) - D'$. As D is γ^e -set of G , there exists a vertex say $u_p \in D$ such that u_t is adjacent to u_p and $|d_G(u_t) - d_G(u_p)| \leq 1$. Hence, in $G \square H$, (u_t, v_k) is adjacent to (u_p, v_k) . Also, $d_{G \square H}(x, y) = d_G(x) + d_H(y)$, where $x \in V(G)$ and $y \in V(H)$. Therefore,

$$\begin{aligned} |d_{G \square H}(u_t, v_k) - d_{G \square H}(u_p, v_k)| &= |d_G(u_t) + d_H(v_k) - (d_G(u_p) + d_H(v_k))| \\ &= |d_G(u_t) - d_G(u_p)| \leq 1. \end{aligned}$$

Thus, D' is an equitable dominating set of $G \square H$, establishing our claim. Hence,

$$\gamma^e(G \square H) \leq |V(H)|\gamma^e(G).$$

As cartesian product of graphs is commutative we get

$$\gamma^e(G \square H) \leq |V(G)|\gamma^e(H).$$

Combining the previous two inequalities we get

$$\gamma^e(G \square H) \leq \min\{|V(H)|\gamma^e(G), |V(G)|\gamma^e(H)\}. \quad \square$$

Illustration 3.2. The upper bound established in the Theorem 3.1 is sharp. Figure 4 illustrates this.

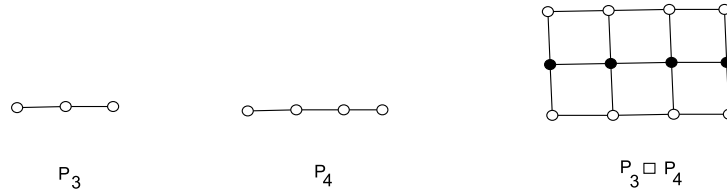


FIGURE 4.

Here, $\gamma^e(P_3) = 1$, $\gamma^e(P_4) = 2$. Darkened vertices forms a γ^e -set of $P_3 \square P_4$ giving $\gamma^e(P_3 \square P_4) = 4$.

Theorem 3.3. $\gamma^e(G \square H) = 1$ if and only if $G = K_1$ and $\gamma^e(H) = 1$.

Proof. Suppose that $\gamma^e(G \square H) = 1$. Then, $G \square H$ has a universal vertex. This is possible only if at least one of the two graphs G, H is K_1 . Let $G = K_1$. Then, $G \square H = H$. Therefore, $\gamma^e(G \square H) = 1$ gives $\gamma^e(H) = 1$. Conversely suppose that $G = K_1$ and $\gamma^e(H) = 1$. Then, $G \square H = H$ giving $\gamma^e(G \square H) = \gamma^e(H) = 1$. $\square$

Lemma 3.4. Suppose G be a graph with at least two universal vertices and either $d_G(u) = |V(G)| - 1$ or $d_G(u) = |V(G)| - 2$ for all $u \in V(G)$, then $\gamma^e(G \square K_2) = 2$.

Proof. Let $V(G) = \{u_1, u_2, u_3, \dots, u_n\}$ and $V(K_2) = \{v_1, v_2\}$. Let u_s, u_t denote the universal vertex in G . Let $D = \{(u_s, v_1), (u_t, v_2)\}$.

Claim : D is a γ^e -set of $G \square K_2$.

From the definition of cartesian product, we get

$$d_{G \square K_2}(u_s, v_1) = d_{G \square H}(u_t, v_2) = n - 1 + 1 = n. \quad (3.1)$$

Let $(u_i, v_j) \in V(G \square K_2) - D$. Then, if $j = 1$, (u_i, v_j) is adjacent to (u_s, v_1) and if $j = 2$, (u_i, v_j) is adjacent to (u_t, v_2) . As $d_G(u) \in \{n - 1, n - 2\}$ for all $u \in V(G)$, we get that

$$d_{G \square K_2}(u_i, v_j) \in \{n, n - 1\}. \quad (3.2)$$

Thus, from (3.1) and (3.2),

$$|d_{G \square K_2}(u_i, v_1) - d_{G \square K_2}(u_s, v_1)| \leq 1$$

and

$$|d_{G \square K_2}(u_i, v_2) - d_{G \square K_2}(u_t, v_2)| \leq 1.$$

Thus, $D = \{(u_s, v_1), (u_t, v_2)\}$ is a γ^e - set of $G \square K_2$ as $\gamma^e(G \square K_2) \geq 2$. As a conclusion we get that $\gamma^e(G \square K_2) = 2$. $\square$

Lemma 3.5. Suppose G be a graph with at least two nonadjacent vertices of degree $|V(G)| - 2$ along with $|V(G)| - 3 \leq d_G(u) \leq |V(G)| - 1$ for all $u \in V(G)$, then $\gamma^e(G \square K_2) = 2$.

Proof. Suppose that the condition holds. Let $V(G) = \{u_1, u_2, u_3, \dots, u_n\}$ and $V(K_2) = \{v_1, v_2\}$. Then by our assumption, $d_G(u) \in \{|V(G)| - 1, |V(G)| - 2, |V(G)| - 3\}$ for all $u \in V(G)$. Suppose u_l, u_m are two nonadjacent vertices in G with degree $|V(G)| - 2$. Let $D = \{(u_l, v_1), (u_m, v_2)\}$.

Claim : D is a γ^e - set of $G \square K_2$.

Let $(u_i, v_j) \in V(G \square K_2) - D$. From the definition of cartesian product, we get that

$$d_{G \square K_2}(u_l, v_1) = d_{G \square H}(u_m, v_2) = n - 2 + 1 = n - 1. \quad (3.3)$$

As $D_G(u_l) = |V(G)| - 2$ and u_m is not adjacent to u_l in G , u_l is adjacent to all the vertices of G except u_m . Thus, u_l is adjacent to u_i . Similarly we can prove that u_m is adjacent to u_k . If $j = 1$, (u_i, v_j) is adjacent to (u_l, v_1) and if $j = 2$, (u_i, v_j) is adjacent to (u_m, v_2) . As $d_G(u) \in \{n - 1, n - 2, n - 3\}$ for all $u \in V(G)$, we get that

$$d_{G \square H}(u_i, v_j) \in \{n, n - 1, n - 2\}. \quad (3.4)$$

Thus, (3.3) and (3.4) gives,

$$|d_{G \square K_2}(u_i, v_1) - d_{G \square K_2}(u_l, v_1)| \leq 1$$

and

$$|d_{G \square K_2}(u_i, v_2) - d_{G \square K_2}(u_m, v_2)| \leq 1.$$

Thus, D is a γ^e - set of $G \square K_2$ as $\gamma^e(G \square K_2) \geq 2$. As a conclusion, we get that $\gamma^e(G \square K_2) = 2$. $\square$

Theorem 3.6. Let G, H be two graphs with at least two vertices. Then, $\gamma^e(G \square H) = 2$ if and only if any one of the following conditions hold.

- (1) $\gamma^e(G) = 1$ and $H = K_2$.
- (2) $H = K_2$ and G have at least two universal vertices along with either $d_G(u) = |V(G)| - 1$ or $d_G(u) = |V(G)| - 2$ for all $u \in V(G)$.
- (3) $H = K_2$ and G have at least two nonadjacent vertices of degree $|V(G)| - 2$ along with $|V(G)| - 3 \leq d_G(u) \leq |V(G)| - 1$ for all $u \in V(G)$.

Proof. Suppose that $\gamma^e(G \square H) = 2$. From [7],

$$\gamma(G \square H) \geq \min\{|V(G)|, |V(H)|\}.$$

Also, $\gamma(G \square H) \leq \gamma^e(G \square H)$. Thus, we get

$$\min\{|V(G)|, |V(H)|\} \leq \gamma^e(G \square H),$$

i.e.,

$$\min\{|V(G)|, |V(H)|\} \leq 2.$$

Thus, at least one of the two graphs G, H contain at most two vertices. This then implies that at least one of the two graphs G, H have exactly two vertices as $|V(G)| \geq 2$ and $|V(H)| \geq 2$. Suppose that H have exactly two vertices. If $|V(G)| = 2$, then $G = K_2$ and hence, $\gamma^e(G) = 1$ giving condition (1) in the theorem statement. Suppose that $|V(G)| \geq 3$. Let $V(G) = \{u_1, u_2, u_3, \dots, u_n\}$ and $V(H) = \{v_1, v_2\}$. Let D be a γ^e - set of $G \square H$. Then, D is in one of the following form.

- (a) $D = \{(u_j, v_1), (u_k, v_1)\}, j \neq k,$
- (b) $D = \{(u_j, v_1), (u_j, v_2)\},$
- (c) $D = \{(u_j, v_1), (u_k, v_2)\}, j \neq k.$

Case (a) : $D = \{(u_j, v_1), (u_k, v_1)\}, j \neq k.$

As $|V(G)| \geq 3$, there exists at least one vertex in say u_s other than u_j and u_k . But then the vertex (u_s, v_2) in $G \square H$ is not dominated by D which is a contradiction to the fact that D is a γ^e - set of $G \square H$. Hence, this case does not occur.

Case (b) : $D = \{(u_j, v_1), (u_j, v_2)\}$.

Let u_t be a vertex in G different from u_j . Consider the vertices (u_t, v_1) and (u_t, v_2) in $G \square H$. As D is a γ^e - set of $G \square H$, the vertex (u_t, v_1) is adjacent to the vertex (u_j, v_1) and

$$|d_{G \square H}(u_t, v_1) - d_{G \square H}(u_j, v_1)| \leq 1. \quad (3.5)$$

Therefore, u_t is adjacent to u_j in G . For every $x \in V(G)$ and $y \in V(H)$

$$d_{G \square H}(x, y) = d_G(x) + d_H(y). \quad (3.6)$$

Applying (3.6) in (3.5), we get

$$|d_G(u_t) - d_G(u_j)| \leq 1.$$

Thus, $\{u_j\}$ is an equitable dominating set of G and hence $\gamma^e(G) = 1$.

Case (c) : $D = \{(u_j, v_1), (u_k, v_2)\}$, $j \neq k$.

As D is a γ^e - set of $G \square H$, all vertices of the form (u_s, v_1) , $s \neq j$ except possibly (u_k, v_1) , is adjacent to (u_j, v_1) with

$$|d_{G \square H}(u_s, v_1) - d_{G \square H}(u_j, v_1)| \leq 1. \quad (3.7)$$

Suppose that (u_k, v_1) is adjacent to (u_j, v_1) in $G \square H$ which implies that u_k is adjacent to u_j in G . Thus, u_j is adjacent to all other vertices in G . Hence, $d_G(u_j) = n - 1$. Similarly we can say that $d_G(u_k) = n - 1$. Thus u_j and u_k are two universal vertices in G . From (3.6), we get that

$$d_{G \square H}(u_j, v_1) = d_{G \square H}(u_k, v_2) = n - 1 + 1 = n.$$

Thus, (3.7) gives

$$|d_{G \square H}(u_s, v_1) - n| \leq 1.$$

Therefore, $d_{G \square H}(u_s, v_1) \in \{n - 1, n, n + 1\}$. $d_{G \square H}(u_s, v_1) \neq n + 1$, because if it does then G contains parallel edges which is a contradiction. Thus, $d_{G \square H}(u_s, v_1) \in \{n - 1, n\}$. By applying (3.6) in $d_{G \square H}(u_s, v_1)$, we get that $d_G(u_s) \in \{n - 2, n - 1\}$. Thus, we have established condition 2 in the theorem statement.

Suppose that (u_k, v_1) is not adjacent to (u_j, v_1) in $G \square H$. u_k is not adjacent to u_j in G which implies that $d_G(u_j) = n - 2$. Similarly $d_G(u_k) = n - 2$. Also,

$$d_{G \square H}(u_j, v_1) = d_G(u_j) + d_H(v_1) = n - 2 + 1 = n - 1.$$

Consider a vertex (u_s, v_1) in $G \square H$ where $s \neq j, k$. Then, as D is a γ^e - set of $G \square H$, the vertex (u_s, v_1) is adjacent to the vertex (u_j, v_1) and

$$|d_{G \square H}(u_s, v_1) - d_{G \square H}(u_j, v_1)| \leq 1,$$

i.e.,

$$|d_{G \square H}(u_s, v_1) - (n - 1)| \leq 1.$$

Therefore, $d_{G \square H}(u_s, v_1) \in \{n - 2, n - 1, n\}$. By applying (3.6) in $d_{G \square H}(u_s, v_1)$, we get that $d_G(u_s) \in \{n - 3, n - 2, n - 1\}$. Thus, we have established condition 3 in the theorem statement.

Conversely suppose that condition (1) holds. Let $\gamma^e(G) = 1$ and $H = K_2$. Clearly $\gamma^e(G \square H) \geq 2$ as there exists no universal vertex in $G \square H$. Let $D' = \{u\}$ be a γ^e - set of G . Therefore, each vertex u' of G is adjacent to u and $|d_G(u) - d_G(u')| \leq 1$. Let $V(H) = \{v_1, v_2\}$. Let $T = \{(u, v_1), (u, v_2)\}$.

Claim : T is an equitable dominating set of $G \square H$.

Let $(u', v_j) \in V(G \square H) - T$, $j \in \{1, 2\}$. Then, $(u, v_j) \in T$ and (u', v_j) is adjacent to (u, v_j) in $G \square H$ as u is adjacent to u' in G . Hence, by applying (3.6) we get that

$$|d_{G \square H}(u, v_j) - d_{G \square H}(u', v_j)| = |d_G(u) - d_G(u')| \leq 1.$$

Thus, T is an equitable dominating set of $G \square H$ giving $\gamma^e(G \square H) \leq 2$. Thus, $\gamma^e(G \square H) = 2$.

Suppose that condition (2) holds. Then, from lemma 3.4, we get $\gamma^e(G \square H) = 2$.

Suppose that condition (3) holds. Then, from lemma 3.5, we get $\gamma^e(G \square H) = 2$. □

Theorem 3.7. Let G be a regular graph on m vertices and P_n the path on n vertices. Then, $\gamma(G) \lceil \frac{n}{3} \rceil \leq \gamma^e(G \boxtimes P_n) \leq \gamma(G) \lceil \frac{n-2}{3} \rceil + 2\gamma(G)$.

Proof. Let $V(G) = \{u_1, u_2, u_3, \dots, u_m\}$ and $P_n = v_1, v_2, v_3, \dots, v_n$. Let G_i denote the i -th G -layer in $G \boxtimes P_n$, with the v_i chosen as the fixed vertex from P_n , $i = 1, 2, \dots, n$. Let $Q_n = v_2, v_3, \dots, v_{n-1}$ denote the path of length $n - 2$ obtained by deleting the vertices v_1 and v_n from the path P_n . Then, $\gamma^e(Q_n) = \lceil \frac{n-2}{3} \rceil$. Let T be a γ^e -set of Q_n and S be a γ -set of G . Let $I = \{i \in \mathbb{N} : u_i \in S\}$ and $J = \{j \in \mathbb{N} : v_j \in T\}$. Denote D as the set $S \times \{v_1\} \cup (S \times T) \cup S \times \{v_2\}$. Then, $|D| = 2|S| + |S||T| = 2\gamma(G) + \gamma(G)\lceil \frac{n-2}{3} \rceil$.

Claim : D is an equitable dominating set of $G \boxtimes P_n$.

Let $(u_i, v_j) \in V(G \boxtimes P_n) - D$. For any two graphs G, H

$$d_{G \boxtimes H}(x, y) = d_G(x) + d_H(y) + d_G(x)d_H(y) \quad \text{where } x \in V(G) \text{ and } y \in V(H). \quad (3.8)$$

Case 1: $i \in I$ and $j \notin J \cup \{1, 2\}$.

As T is a γ^e -set of Q_n , there exists a vertex v_k in T such that v_k is adjacent to v_j . Hence, $(u_i, v_k) \in D$ and (u_i, v_k) is adjacent to (u_i, v_j) in $G \boxtimes P_n$. From (3.8),

$$|d_{G \boxtimes P_n}(u_i, v_k) - d_{G \boxtimes P_n}(u_i, v_j)| = |(d_{P_n}(v_k) - d_{P_n}(v_j))(1 + d_G(u_i))|. \quad (3.9)$$

But $d_{P_n}(v_k) = d_{P_n}(v_j) = 2$. Thus, from (3.9),

$$\text{Case 2: } i \notin I \text{ and } j \in J. \quad |d_{G \boxtimes P_n}(u_i, v_k) - d_{G \boxtimes P_n}(u_i, v_j)| = 0.$$

As S is a γ^e -set of G , there exists a vertex u_l in G such that u_l is adjacent to u_i . Hence, $(u_l, v_j) \in D$ and (u_l, v_j) is adjacent to (u_i, v_j) in $G \boxtimes P_n$. (3.8) gives

$$|d_{G \boxtimes P_n}(u_l, v_j) - d_{G \boxtimes P_n}(u_i, v_j)| = |(d_G(u_l) - d_G(u_i))(1 + d_{P_n}(v_j))|. \quad (3.10)$$

As G is a regular graph $d_G(u_l) = d_G(u_i)$. Thus, from (3.10),

$$|d_{G \boxtimes P_n}(u_l, v_j) - d_{G \boxtimes P_n}(u_i, v_j)| = 0.$$

Case 3: $i \notin I$ and $v_j \in \{v_1, v_2\}$.

Let $v_j = v_1$. As S is a γ^e -set of G , there exists a vertex u_s in G such that u_s is adjacent to u_i . Hence, $(u_s, v_1) \in D$ and (u_s, v_1) is adjacent to (u_i, v_1) in $G \boxtimes P_n$. From (3.8),

$$|d_{G \boxtimes P_n}(u_s, v_1) - d_{G \boxtimes P_n}(u_i, v_1)| = |(d_G(u_s) - d_G(u_i))(1 + d_{P_n}(v_1))|. \quad (3.11)$$

As G is a regular graph $d_G(u_s) = d_G(u_i)$. Thus, from (3.11),

$$|d_{G \boxtimes P_n}(u_s, v_1) - d_{G \boxtimes P_n}(u_i, v_1)| = 0.$$

Case 4: $i \notin I$ and $j \notin J \cup \{v_1, v_2\}$.

As S is a γ^e -set of G , there exists a vertex u_t in G such that u_t is adjacent to u_i . As T is a γ^e -set of Q_n , there exists a vertex v_p in T such that v_p is adjacent to v_j . Hence, $(u_t, v_p) \in D$ and (u_t, v_p) is adjacent to (u_i, v_1) in $G \boxtimes P_n$. From (3.8), $|d_{G \boxtimes P_n}(u_t, v_p) - d_{G \boxtimes P_n}(u_i, v_j)|$ equals

$$|(d_G(u_t) - d_G(u_i)) + (d_{P_n}(v_p) - d_{P_n}(v_j)) + d_G(u_t)d_{P_n}(v_p) - d_G(u_i)d_{P_n}(v_j)|. \quad (3.12)$$

As G is a regular graph $d_G(u_t) = d_G(u_i)$. Also $d_{P_n}(v_p) = d_{P_n}(v_j) = 2$. Thus, from (3.12),

$$|d_{G \boxtimes P_n}(u_t, v_p) - d_{G \boxtimes P_n}(u_i, v_j)| = 0.$$

Thus, D is an equitable dominating set of $G \boxtimes P_n$, establishing our claim. Hence,

$$\gamma^e(G \boxtimes P_n) \leq \gamma(G)\lceil \frac{n-2}{3} \rceil + 2\gamma(G). \quad (3.13)$$

To obtain the lower bound, let D' be a γ^e -set of $G \boxtimes P_n$. Let $U = \{u_i : (u_i, v_j) \in D' \text{ for some vertex } v_j \text{ in } P_n\}$ and let $V = \{v_j : (u_i, v_j) \in D' \text{ for some vertex } u_i \text{ in } G\}$

Claim : U, V forms a dominating set of G and P_n respectively.

Let $u_s \in V(G) - U$. Let v_k be any arbitrarily chosen vertex of P_n . Then, the vertex $(u_s, v_k) \notin D'$. As D' is a γ^e -set of $G \boxtimes P_n$, there exists a vertex $(u_t, v_x) \in D'$ such that (u_s, v_k) is adjacent to (u_t, v_x) in $G \boxtimes P_n$ and

$$|d_{G \boxtimes P_n}(u_s, v_k) - d_{G \boxtimes P_n}(u_t, v_x)| \leq 1.$$

Hence, $u_t \in U$ and u_t is adjacent to u_s in G . Therefore U is a dominating set of G giving

$$|U| \geq \gamma(G).$$

To prove that V forms a dominating set of P_n , let $v_r \in V(P_n) - V$. Let u_q be any arbitrarily chosen vertex of G . Then, the vertex $(u_q, v_r) \notin D'$. As D' is a γ^e -set of $G \boxtimes P_n$, there exists a vertex $(u_y, v_z) \in D'$ such that (u_q, v_r) is adjacent to (u_y, v_z) in $G \boxtimes P_n$ and

$$|d_{G \boxtimes P_n}(u_q, v_r) - d_{G \boxtimes P_n}(u_y, v_z)| \leq 1.$$

Hence, $v_z \in U$ and from the definition of strong product v_r is adjacent to v_z in P_n . Therefore, V is a dominating set of P_n giving $|V| \geq \gamma(P_n) = \lceil \frac{n}{3} \rceil$.

As $D' = U \times V$, we get

$$|D'| = |U||V| \geq \gamma(G) \lceil \frac{n}{3} \rceil. \quad (3.14)$$

Combining (3.13) and (3.14), we get

$$\gamma(G) \lceil \frac{n}{3} \rceil \leq \gamma^e(G \boxtimes P_n) \leq \gamma(G) \lceil \frac{n-2}{3} \rceil + 2\gamma(G). \quad \square$$

4. CONCLUSION

Our primary contributions include establishing upper bounds for the equitable and outdegree equitable domination numbers of the Cartesian product of graphs along with the characterisations for these domination numbers for the particular cases. The paper further presented an upper bound for the outdegree equitable domination number of the strong product of special types of regular graphs. We also provided bounds for the equitable domination number of the strong product of a graph and a path. The findings advance the study of equitable and out-degree equitable domination in graph theory creating a foundation for future research on other graph products.

CONFLICTS OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this article.

AUTHORS CONTRIBUTION STATEMENT

All authors contributed equally to this paper and have reviewed and approved the final version of the manuscript.

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