



The Generalized Harmonic Numbers with Parameters δ and γ , $H_{l,k}(\delta, \gamma)$, and Their Applications

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ABSTRACT. In this paper, we define generalized harmonic numbers with parameters δ and γ , $H_{l,k}(\delta, \gamma)$. As their implementation, we obtain some results and combinatorial identities by using the partial derivative and generating functions. Additionally, we give graphs of the roots of $H_{l,k}(\delta, \gamma) = 0$, $G_l(\delta, \gamma) = 0$ and $G_l^{(k)}(\delta, \gamma) = 0$ for different parameters. Thus, we have made the motivation and roots of the given equations more understandable. In particular, with this approach, we have improved the understanding of their structural properties and revealed new insights for further research.

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1. INTRODUCTION

The gamma function serves as a prevalent extension of the factorial function to complex numbers. Its definition spans all complex numbers except for those that are non-positive integers. Crafted by Daniel Bernoulli, the gamma function finds its definition for complex numbers exhibiting a positive real component, established through the convergence of an improper integral. The gamma function is obtained by for $\nu \in \mathbb{C}$ with $\Re(\nu) > 0$,

$$\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt,$$

with $\Gamma(\nu + 1) = \nu\Gamma(\nu)$ and $\Gamma(1) = 1$ [25, 30]. The binomial coefficients can be expressed as gamma function

$$\binom{\nu}{w} = \frac{\Gamma(\nu + 1)}{\Gamma(w + 1)\Gamma(\nu - w + 1)} \text{ for } \nu, w \in \mathbb{C} \setminus \mathbb{Z}^-.$$

Within the realm of mathematics, the Lerch zeta function, occasionally referred to as the Hurwitz–Lerch zeta function, stands as a unique mathematical construct that serves as a generalization of both the Hurwitz zeta function and the polylogarithm. This function, bearing the name of Czech mathematician Mathias Lerch, was brought to light through his

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publication in 1887, shedding further light on its intricate properties and connections within mathematical frameworks. A general Hurwitz-Lerch zeta function $\Phi_\kappa(\delta, \gamma)$ [10] is stated in the following

$$\Phi_\kappa(\delta, \gamma) = \sum_{h=0}^{\infty} \frac{\gamma^h}{(\hbar + \delta)^\kappa},$$

where $\delta \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $\kappa \in \mathbb{C}$ for $|\gamma| < 1$ and $\Re(\kappa) > 1$ for $|\gamma| = 1$. The integral form of $\Phi_\kappa(\delta, \gamma)$ [10, 29] is presented by

$$\Phi_\kappa(\delta, \gamma) = \frac{1}{\Gamma(\kappa)} \int_0^\infty \frac{t^{\kappa-1} e^{-\delta t}}{1 - \gamma e^{-t}} dt,$$

where $\Re(\delta) > 0$, $\Re(\gamma) > 0$ when $|\gamma| \leq 1$ ($\gamma \neq 1$); $\Re(\kappa) > 1$ when $\gamma = 1$. As its particular instances, the Riemann zeta is stated by

$$\Phi_\kappa(0, 1) = \zeta(\kappa) = \sum_{h=0}^{\infty} \frac{1}{\hbar^\kappa} = \prod_{p \text{ prime}} (1 - p^{-\kappa})^{-1},$$

where κ is a complex variable with $\Re(\kappa) > 1$. For example, taking $\kappa = 1$, $\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \dots$, which could be proven to diverge, and for $\kappa = 2$, the well-known fixed [11, 31] is $\zeta(2) = \pi^2/6$. This particular function found its inception within the realm of complex analysis back in 1859, garnering significant significance owing to its profound interrelation with various other realms of mathematics, such as combinatorics and matrix theory, all stemming from its tie-in with the Riemann hypothesis. The appellation of the Hurwitz zeta function traces back to its originator, Adolf Hurwitz, who brought it into mathematical discourse in the year 1882. Generally, the Hurwitz zeta function is stated by for complex variable κ with $\Re(\kappa) > 1$ and $\delta > 0$,

$$\Phi_\kappa(\delta, 1) = \zeta(\delta, \kappa) = \sum_{h=0}^{\infty} \frac{1}{(\hbar + \delta)^\kappa}.$$

The exploration of harmonic numbers dates back to ancient times, marking their significance across diverse domains within number theory. Often colloquially referred to as harmonic series, they maintain a close association with the Riemann zeta function and frequently emerge within the formulations of numerous specialized functions. Notably, harmonic numbers serve as an approximation to the natural logarithm function, reflecting the characteristic that the corresponding harmonic series exhibits unbounded growth, albeit at a gradual pace. The harmonic numbers, showed by H_l , are indicated by for $l \geq 1$

$$H_0 = 0 \text{ and } H_l = \sum_{\wp=1}^l \frac{1}{\wp},$$

and their generating function is

$$\frac{-\ln(1-t)}{1-t} = \sum_{h=0}^{\infty} H_h t^h.$$

For quite some time, the authors have explored harmonic numbers, which hold great significance across various domains of number theory. In diverse research fields, these numbers are considered [12, 13, 23, 28]. There are numerous studies revolving around harmonic numbers and generalized of them [4–6, 9, 12, 26, 27]. Guo and Cha [15] stated the generalized harmonic numbers by for $l \geq 1$

$$H_l(\sigma) = \sum_{\wp=1}^l \frac{\sigma^\wp}{\wp},$$

where $H_0(\sigma) = 0$, σ represents a suitable parameter. Their generating function is

$$\frac{-\ln(1-\sigma t)}{1-t} = \sum_{h=0}^{\infty} H_h(\sigma) t^h. \quad (1.1)$$

For $\alpha \in \mathbb{R}^+$, taking $\sigma = 1/\alpha$, $H_l(1/\alpha)$ are called the generalized harmonic numbers [12].

In [3], Cheon et al. defined the generalized harmonic numbers by for $\hbar, r \geq 1$,

$$H\langle \hbar, 0 \rangle = 1 \text{ and } H\langle \hbar, r \rangle = \sum_{1 \leq j_1 < j_2 < \dots < j_r \leq \hbar} \frac{1}{j_1 j_2 \dots j_r}. \quad (1.2)$$

The numbers $H\langle\hbar, r\rangle$ satisfy the recurrence relation as follows

$$H\langle\hbar, r\rangle = H\langle\hbar - 1, r\rangle + \frac{1}{\hbar}H\langle\hbar - 1, r - 1\rangle.$$

The nomenclature "Beta function" is attributed to Legendre as well as Whittaker and Watson in their work from 1990, denoting what is commonly known as the Beta integral, which is also referred to as the Eulerian integral of the first kind. The Beta function [1, 30] is defined by for $\Re(\nu > 0)$, $\Re(w > 0)$,

$$B(\nu, w) = \int_0^1 t^{\nu-1} (1-t)^{w-1} dt.$$

Clearly, $B(\nu, w) = \frac{\Gamma(\nu)\Gamma(w)}{\Gamma(\nu+w)}$ ([30]).

The inception of Fibonacci numbers dates back to as early as 200 BC in Indian mathematical texts. The name "Fibonacci numbers" is attributed to the Italian mathematician Leonardo of Pisa, or Fibonacci, who introduced this sequence to the mathematical discourse of Western Europe through his seminal work, *Liber Abaci*, published in 1202. The Fibonacci sequence is characterized by a number being equal to the sum of the two consecutive numbers preceding it. The Fibonacci sequence $\{F_m\}$ is expressed by the recursion relation for $m > 1$, $F_{m+1} = F_m + F_{m-1}$ with the initial conditions 0 and 1. The Binet formula is $F_m = (\alpha^m - \beta^m) / (\alpha - \beta)$, where $\alpha, \beta = (1 \pm \sqrt{5})/2$. This sequence unfolds as follows:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots$$

For $l, \kappa \in \mathbb{N}$ and an variable δ , the authors [5, 7] defined the generalized harmonic function by

$$H_{l,\kappa}(\delta) = \sum_{\wp=1}^l \frac{1}{(\wp + \delta + 1)^\kappa}.$$

There have been many recent studies in the literature on these numbers [8, 16–22]. For $\delta = 0$, note that $H_{l,\kappa}(0)$ is the generalized harmonic number of order κ . Through the utilization of several definite integrals, Kim et al. [16] formulated both a finite summation expression and an infinite summation expression, encompassing harmonic numbers up to a specified order. In [17], Kim et al. establish several identities connecting generalized harmonic numbers $H_{l,\kappa}(0)$ and generalized harmonic functions $H_{l,\kappa}(\delta)$ by employing fundamental methods such as integration, differentiation, and Binomial Theorem. Using $G_l(\delta) = B(\delta + 1, l + 1)$, ($l \in \mathbb{N}$), for every $\kappa \in \{2, 3, 4, 5\}$,

$$\sum_{\wp=0}^l \binom{l}{\wp} \frac{(-1)^\wp}{(\wp + \delta + 1)^\kappa},$$

are entirely represented using generalized harmonic numbers and generalized harmonic functions. Also, they showed that

$$\sum_{\wp=0}^l (-1)^\wp \binom{l}{\wp} \binom{\delta + \wp + 1}{\wp}^{-1} H_{\wp,1}(\delta) = \frac{\delta + 1}{(l + \delta + 1)^2}.$$

From here, for $l \geq 0$,

$$\begin{aligned} \sum_{\wp=0}^l \binom{l}{\wp} \frac{(-1)^\wp}{(\wp + \delta + 1)^2} &= \frac{H_{l,1}(\delta)}{\binom{\delta+l+1}{l}(\delta+1)}, \\ \sum_{\wp=0}^l \binom{l}{\wp} \frac{(-1)^\wp}{(\wp + \delta + 1)^3} &= \frac{H_{l,2}(\delta) + (H_{l,1}(\delta))^2}{2! \binom{\delta+l+1}{l}(\delta+1)}, \end{aligned} \tag{1.3}$$

and

$$\sum_{\wp=0}^l \binom{l}{\wp} \frac{(-1)^\wp}{(\wp + \delta + 1)^4} = \frac{2H_{l,3}(\delta) + 3H_{l,1}(\delta)H_{l,2}(\delta) + (H_{l,1}(\delta))^3}{3! \binom{\delta+l+1}{l}(\delta+1)}.$$

In [2], Batir proved a new combinatorial identity and applying it, he established many finite harmonic sum identities. For example, for $l \geq 0$,

$$\sum_{\wp=0}^l \frac{(-1)^\wp}{\wp} \left(\binom{l}{\wp} H_{l-\wp} + \frac{1}{\wp} \binom{l}{\wp}^{-1} \right) = -H_l^2.$$

In [13], Guo showed an elementary proof for a binomial identity of Batir [2], several combinatorial identities pertaining to harmonic numbers and binomial coefficients. For example, for $l \geq 0$,

$$\sum_{\wp=0}^{l-1} \frac{\binom{\delta+l+1}{\wp}}{(l-\wp)\binom{l}{\wp}} H_\wp = \frac{H_l^2 - H_{l,2}(0)}{2} + \sum_{\wp=1}^l \frac{\binom{\delta+\wp}{\wp}}{\wp^2} (\wp H_l - 1).$$

The Cauchy numbers, C_h , are stated by the generating functions to be

$$\sum_{h=0}^{\infty} C_h \frac{t^h}{h!} = \frac{t}{\ln(1+t)}. \quad (1.4)$$

The generating function in terms of exponentials is as follows

$$\sum_{h=0}^{\infty} \frac{t^h}{h!} = e^t. \quad (1.5)$$

Stirling numbers find their relevance across a spectrum of both analytical and combinatorial challenges. They owe their name to James Stirling, who originally introduced them within the realm of algebraic exploration in his publication, "Methodus differentialis" (1730). Despite their initial algebraic context, Masanobu Saka revitalized these numbers in 1782, attributing them with a combinatorial significance. There exist two distinct categories of Stirling numbers: those of the first kind and those of the second kind. What unites these sets is their role in delineating coefficients that establish connections between disparate sequences of polynomials, frequently encountered within combinatorial studies. Furthermore, each set can be interpreted as denoting the count of partitions formed from n elements, distributed into k non-empty subsets, each imbued with a specific form of order (be it unordered, cyclical, or linear). The Stirling numbers of the first kind $S_1(\tilde{h}, \wp)$ are indicated by

$$w^{\tilde{h}} = \sum_{\wp=0}^{\tilde{h}} S_1(\tilde{h}, \wp) w^\wp,$$

and Stirling numbers of the second kind $S_2(\tilde{h}, \wp)$ are indicated by

$$w^{\tilde{h}} = \sum_{\wp=0}^{\tilde{h}} S_2(\tilde{h}, \wp) w^\wp,$$

where $w^{\tilde{h}}$ stands for the falling factorial is highlighted with $w^0 = 1$ and $w^{\tilde{h}} = \prod_{k=0}^{\tilde{h}-1} (w - k)$. The generating functions of $S_1(\tilde{h}, \wp)$ and $S_2(\tilde{h}, \wp)$, respectively, are specified by for $\wp \geq 0$

$$\sum_{h=\wp}^{\infty} S_1(\tilde{h}, \wp) \frac{t^h}{h!} = \frac{(\ln(1+t))^\wp}{\wp!},$$

and

$$\sum_{h=\wp}^{\infty} S_2(\tilde{h}, \wp) \frac{t^h}{h!} = \frac{(e^t - 1)^\wp}{\wp!}. \quad (1.6)$$

2. THE GENERALIZED HARMONIC NUMBERS WITH PARAMETERS δ AND γ , $H_{l,\kappa}(\delta, \gamma)$, AND THEIR APPLICATIONS

In this section, the motivation for this study is taken from [17]. The generalized harmonic numbers with parameters δ and γ , $H_{l,\kappa}(\delta, \gamma)$, are specified and subsequently presented with fresh findings.

Definition 2.1. For nonnegative integer l and positive integer κ , generalized harmonic numbers with parameters δ and γ are defined by

$$H_{l,\kappa}(\delta, \gamma) = \sum_{\wp=0}^l \frac{\gamma^{\wp}}{(\wp + \delta + 1)^{\kappa}}, \quad (2.1)$$

where $\delta \geq 0$ and $\gamma \setminus \{0\}$ are parameters.

When $\delta = 0$ and $\gamma = 1$ in (2.1), $H_{l,\kappa}(0, 1) = H_{l,\kappa}(0)$.

Some terms of $H_{l,\kappa}(\delta, \gamma)$ are as follows:

$$\begin{aligned} H_{1,2}(3, \gamma) &= \frac{1}{25}\gamma + \frac{1}{16}, \\ H_{1,2}(\delta, 3) &= \frac{4\delta^2 + 10\delta + 7}{(\delta^2 + 3\delta + 2)^2}, \\ H_{2,2}(\delta, 3) &= \frac{13\delta^4 + 88\delta^3 + 220\delta^2 + 240\delta + 99}{(\delta^3 + 6\delta^2 + 11\delta + 6)^2}, \\ H_{5,1}(4, \gamma) &= \frac{1}{10}\gamma^5 + \frac{1}{9}\gamma^4 + \frac{1}{8}\gamma^3 + \frac{1}{7}\gamma^2 + \frac{1}{6}\gamma + \frac{1}{5}, \\ H_{5,2}(4, \gamma) &= \frac{1}{100}\gamma^5 + \frac{1}{81}\gamma^4 + \frac{1}{64}\gamma^3 + \frac{1}{49}\gamma^2 + \frac{1}{36}\gamma + \frac{1}{25}. \end{aligned}$$

To clearly explain the motivation and better understand the roots of the given equations, we provide detailed information and visual representations related to these generalized harmonic numbers. This approach enhances the understanding of their structural properties and reveals new insights for further exploration. The numerical roots of these terms are as follows:

$$\begin{aligned} H_{1,2}(3, \gamma) = 0 &\implies \gamma = -1.5625, \\ H_{1,2}(\delta, 3) = 0 &\implies \delta = -1.25 - 0.43301i, -1.25 + 0.43301i, \\ H_{2,2}(\delta, 3) = 0 &\implies \delta = -1.1386 - 0.39591i, -1.1386 + 0.39591i, \\ &\quad -2.2461 - 0.44294i, -2.2461 + 0.44294i, \\ H_{5,1}(4, \gamma) = 0 &\implies \gamma = -1.1507, -0.56927 + 0.99932i, -0.56927 - 0.99932i, \\ &\quad 0.58904 + 0.98342i, 0.58904 - 0.98342i, \\ H_{5,2}(4, \gamma) = 0 &\implies \gamma = -0.64811 + 1.1528i, -0.64811 - 1.1528i, \\ &\quad 0.69274 + 1.1170i, 0.69274 - 1.1170i, -1.3238. \end{aligned}$$

Now, let's show the roots of some $H_{l,\kappa}(\delta, \gamma) = 0$ on the graphs.

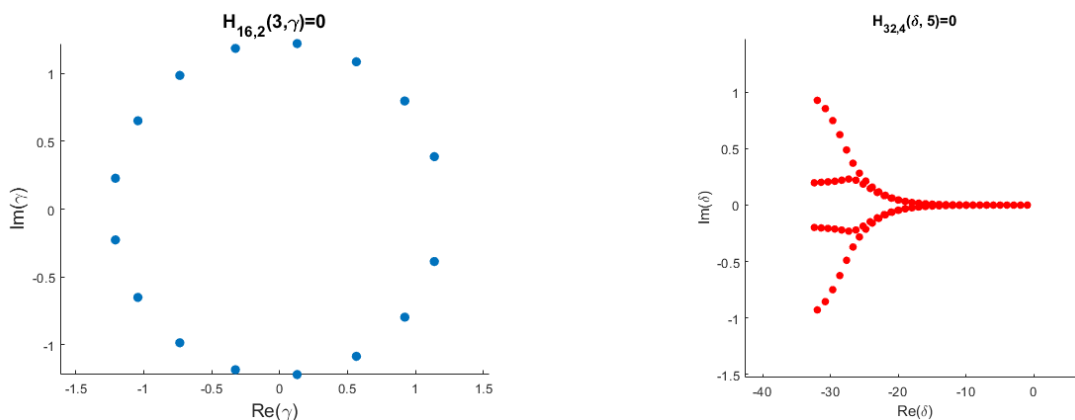


FIGURE 1. Plots of the roots of some $H_{l,\kappa}(\delta, \gamma) = 0$.

The distribution of values in the left graph in Figure 1 is circular. According to the right graph Figure 1, values in the range $Re(\delta) \approx (-15, 0)$ are concentrated around $Im(\delta) \approx 0$. Values in the range approximately $Re(\delta) \approx (-25, -15)$ split into two branches, while values in the range approximately $Re(\delta) \approx (-35, -25)$ split into four branches.

The distributions of the roots of some $H_{l,\kappa}(\delta, \gamma) = 0$ for $1 \leq l \leq 16$ are as follows:

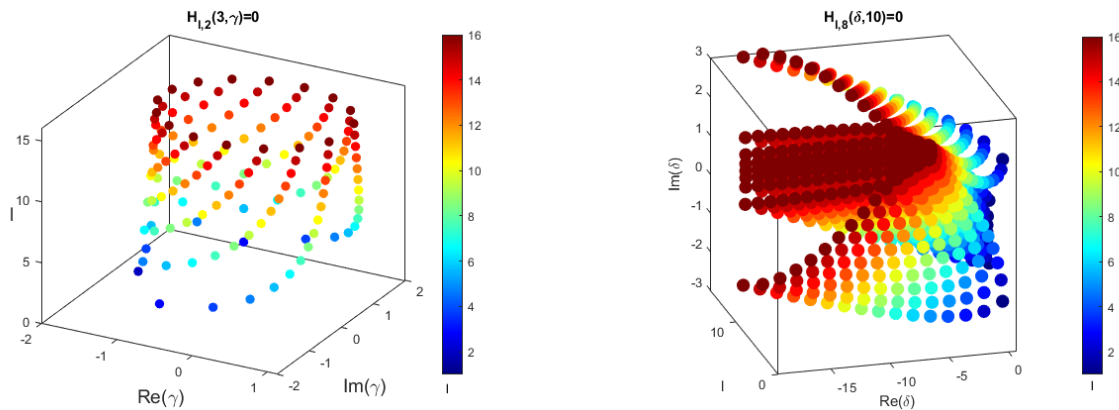


FIGURE 2. Graphs of the roots of some $H_{l,\kappa}(\delta, \gamma) = 0$ for $1 \leq l \leq 16$.

In Figure 2, when the left graph is examined, a clear separation of the curves indicates a parametric spectral solution or root distribution. When the right graph is examined, a curvilinear and wavy structure is seen.

Now, a number with two parameters $G_{l,\kappa}(\delta, \gamma)$, associated with $H_{l,\kappa}(\delta, \gamma)$, will be given. When $\kappa = 1$, this number corresponds to the binomial representation of the number with two parameters $H_l(\delta, \gamma)$ as

$$G_l(\delta, \gamma) = \int_0^1 (1 + \gamma t)^l t^\delta dt = \sum_{\wp=0}^l \binom{l}{\wp} \frac{\gamma^\wp}{\wp + \delta + 1}, \quad (2.2)$$

and also

$$G_{l+1}(\delta, \gamma) = G_l(\delta, \gamma) + \gamma G_l(\delta + 1, \gamma).$$

From here, for $\gamma = -1$, we can write

$$G_l(\delta, -1) = \int_0^1 (1 - t)^l t^\delta dt = \sum_{\wp=0}^l \binom{l}{\wp} \frac{(-1)^\wp}{\wp + \delta + 1} = \frac{1}{\binom{\delta+l+1}{l}(\delta+1)}, \text{ (see [17])}.$$

Some terms of $G_l(\delta, \gamma)$ are as follows:

$$\begin{aligned} G_1(2, \gamma) &= \frac{1}{4}\gamma + \frac{1}{3}, \\ G_1(\delta, 2) &= \frac{3\delta + 4}{\delta^2 + 3\delta + 2}, \\ G_3(15, \gamma) &= \frac{1}{19}\gamma^3 + \frac{1}{6}\gamma^2 + \frac{3}{17}\gamma + \frac{1}{16}, \\ G_3(\delta, 12) &= \frac{2197\delta^3 + 13\,689\delta^2 + 25\,766\delta + 14\,280}{\delta^4 + 10\delta^3 + 35\delta^2 + 50\delta + 24}, \\ G_5(4, \gamma) &= \frac{1}{10}\gamma^5 + \frac{5}{9}\gamma^4 + \frac{5}{4}\gamma^3 + \frac{10}{7}\gamma^2 + \frac{5}{6}\gamma + \frac{1}{5}, \\ G_5(\delta, 4) &= \frac{3125\delta^5 + 50\,000\delta^4 + 299\,375\delta^3 + 829\,000\delta^2 + 1045\,100\delta + 468\,720}{\delta^6 + 21\delta^5 + 175\delta^4 + 735\delta^3 + 1624\delta^2 + 1764\delta + 720}. \end{aligned}$$

To clearly explain the motivation and better understand the roots of the given equations, we provide detailed information and visual representations related to these generalized harmonic numbers. This approach enhances the comprehension

of their structural properties and reveals new insights for further exploration. The numerical roots of these terms are as follows:

$$\begin{aligned} G_1(2, \gamma) = 0 &\implies \gamma = -1.3333, \\ G_1(\delta, 2) = 0 &\implies \delta = -1.3333, \\ G_3(15, \gamma) = 0 &\implies \gamma = -1.0355 - 0.10749i, -1.0355 + 0.10749i, -1.0956, \\ G_3(\delta, 12) = 0 &\implies \delta = -1.0012, -2.0273, -3.2023, \\ G_5(4, \gamma) = 0 &\implies \gamma = -1.2209 - 0.28470i, -1.2209 + 0.28470i, -1.3448, \\ &\quad -0.88452 + 0.40492i, -0.88452 - 0.40492i, \\ G_5(\delta, 4) = 0 &\implies \delta = -1.0012, -2.0171, -4.2843, -5.6040, -3.0935. \end{aligned}$$

Now, let's show the roots of some $G_l(\delta, \gamma) = 0$ on the graphs. The distribution of values in the right graph in Figure 3

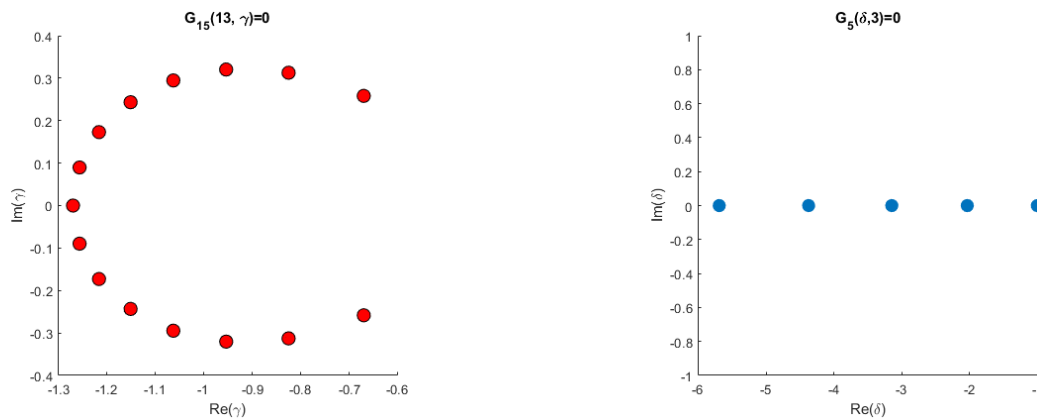


FIGURE 3. Plots of the roots of some $G_l(\delta, \gamma) = 0$.

is linear. The values of the left graph in Figure 3 are distributed in a circular arc (C-shape).

The distributions of the roots of some $G_l(\delta, \gamma) = 0$ for $1 \leq l \leq 16$ are as follows: In Figure 4, the points on the

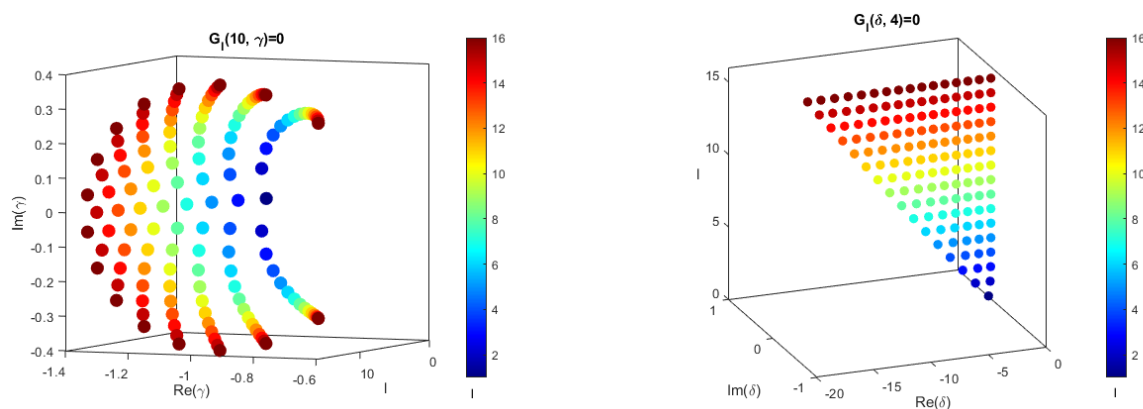


FIGURE 4. Distributions of the roots of some $G_l(\delta, \gamma) = 0$ for $1 \leq l \leq 16$.

left graph form smooth and curvilinear patterns in the real and imaginary components of γ . The points on the left graph form smooth and curvilinear patterns in the real and imaginary components of γ , while the points seen on the right graph are concentrated in a triangular region.

From here, for $\gamma = -1$, we can write

$$G_l(\delta, -1) = \int_0^1 (1-t)^l t^\delta dt = \sum_{\wp=0}^l \binom{l}{\wp} \frac{(-1)^\wp}{\wp + \delta + 1} = \frac{1}{\binom{\delta+l+1}{l}(\delta+1)}, \text{ (see [17]).}$$

Lemma 2.2. Let $G_l(\delta, \gamma)$ be as above. For nonnegative integer l and positive integer κ , it holds that

$$G_l^{(\kappa)}(\delta, \gamma) = (-1)^\kappa \kappa! \sum_{\wp=0}^l \binom{l}{\wp} \frac{\gamma^\wp}{(\wp + \delta + 1)^{\kappa+1}},$$

$$\gamma(l+1)G_l(\delta, \gamma) - (1+\gamma)^{l+1} + \delta G_{l+1}(\delta-1, \gamma) = 0, \quad (2.3)$$

and

$$\gamma(l+1)G_l^{(\kappa)}(\delta, \gamma) + \kappa G_{l+1}^{(\kappa-1)}(\delta-1, \gamma) + \delta G_{l+1}^{(\kappa)}(\delta-1, \gamma) = 0, \quad (2.4)$$

where δ and γ are as above.

Proof. Firstly, let's start by computing the partial derivative with respect to the variable δ at (2.2). Then

$$G_l'(\delta, \gamma) = - \sum_{\wp=0}^l \binom{l}{\wp} \frac{\gamma^\wp}{(\wp + \delta + 1)^2},$$

$$G_l''(\delta, \gamma) = \sum_{\wp=0}^l \binom{l}{\wp} \frac{2\gamma^\wp}{(\wp + \delta + 1)^3}.$$

If this process is continued, the result is obtained by taking κ th order partial derivative. Secondly, using the integration by parts, we have the result. Finally, by iteratively applying κ successive partial derivatives with respect to δ at (2.3), we can have the desired recurrence relation. $\square$

Some terms of $G_l^{(\kappa)}(\delta, \gamma)$ are as follows:

$$G_2^{(1)}(3, \gamma) = \frac{1}{108}\gamma^2 + \frac{4}{125}\gamma + \frac{1}{32},$$

$$G_2^{(1)}(\delta, 2) = \frac{9\delta^4 + 66\delta^3 + 177\delta^2 + 204\delta + 88}{(\delta^3 + 6\delta^2 + 11\delta + 6)^2},$$

$$G_2^{(2)}(\delta, 3) = \frac{4(8\delta^6 + 84\delta^5 + 366\delta^4 + 844\delta^3 + 1089\delta^2 + 756\delta + 225)}{(\delta^3 + 6\delta^2 + 11\delta + 6)^3},$$

$$G_3^{(2)}(5, \gamma) = \frac{2}{729}\gamma^3 + \frac{3}{256}\gamma^2 + \frac{6}{343}\gamma + \frac{1}{108}.$$

To clearly explain the motivation and better understand the roots of the given equations, we provide detailed information and visual representations related to these generalized harmonic numbers. This approach enhances the understanding of their structural properties and reveals new insights for further exploration. The numerical roots of these terms are as follows:

$$G_2^{(1)}(3, \gamma) = 0 \implies \gamma = -1.728 - 0.62371i, -1.728 + 0.62371i,$$

$$G_2^{(1)}(\delta, 2) = 0 \implies \delta = -2.5080 - 0.50991i, -2.5080 + 0.50991i,$$

$$\quad -1.1587 - 0.38757i, -1.1587 + 0.38757i,$$

$$G_2^{(2)}(\delta, 3) = 0 \implies \delta = -2.3619 + 0.86i, -2.3619 - 0.86i, -2.4667,$$

$$\quad -0.98058 + 0.61392i, -0.98058 - 0.61392i,$$

$$\quad -1.3483,$$

$$G_3^{(2)}(5, \gamma) = 0 \implies \gamma = -1.3206 + 0.57115i, -1.3206 - 0.57115i,$$

$$\quad -1.6303.$$

Now, let's show the roots of some $G_l^{(\kappa)}(\delta, \gamma) = 0$ on the graphs. The values of the left graph in Figure 5 are distributed in

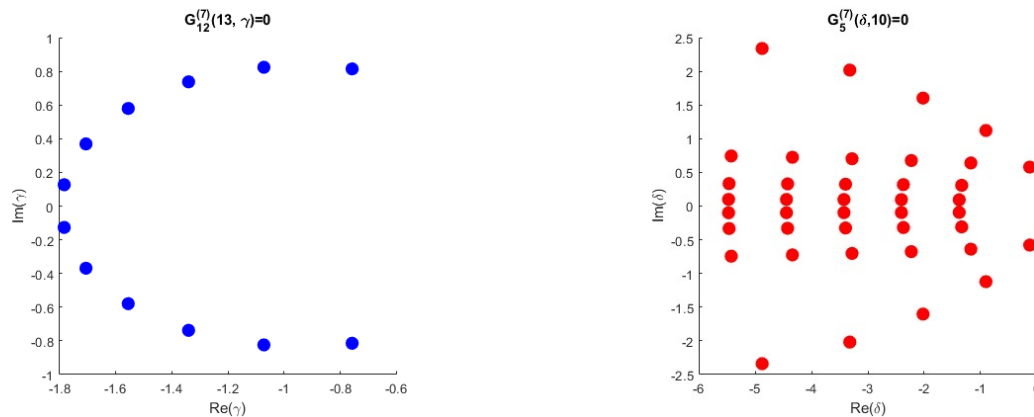


FIGURE 5. Plots of the roots of some $G_l^{(\kappa)}(\delta, \gamma) = 0$.

a parabolic shape. The values of the right graph in Figure 5, the majority of values of $Im(\delta)$ fit into the range $(-0.8, 0.8)$.

The distributions of the roots of some $G_l^{(\kappa)}(\delta, \gamma) = 0$ for $1 \leq l \leq 16$ are as follows: In Figure 6, when the graphs

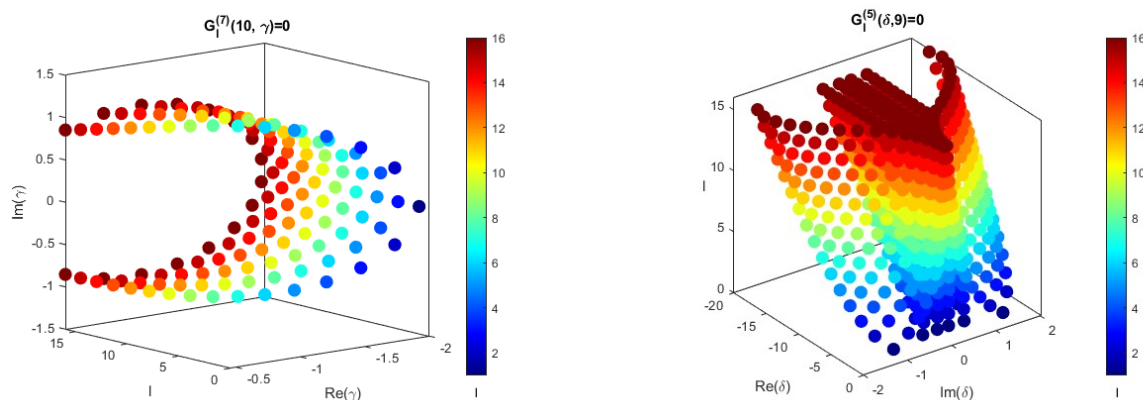


FIGURE 6. Distributions of the roots of some $G_l^{(\kappa)}(\delta, \gamma) = 0$ for $1 \leq l \leq 16$.

are examined, it is seen that the roots are symmetric. In the right graph, the solution seems to be concentrated around $Re(\delta) \approx 0$, with large values of l concentrated at more extreme points, while in the left graph, it seems to be concentrated around $Re(\gamma) \approx 0.5$.

Now, we can give an equality involving $H\langle\delta, \gamma\rangle$ as follows:

Theorem 2.3. For nonnegative integer l and positive integer κ , it holds that

$$G_l^{(\kappa)}(\delta, \gamma) = (-1)^\kappa \kappa! \binom{l+\delta}{l}^{-1} \sum_{\omega=0}^{\delta} \sum_{\wp=0}^{l+\delta} \binom{l+\delta}{\wp} \frac{(-1)^{\omega+\delta} \gamma^{\wp-\delta}}{(\wp+1)^{\kappa-\omega+1}} H\langle\delta, \omega\rangle, \quad (2.5)$$

where δ is positive integer and γ is parameter.

Proof. From the iteration method, by (2.4), we have

$$G_l^{(\kappa)}(\delta, \gamma) = -\frac{1}{\gamma(l+1)} \left(\kappa G_{l+1}^{(\kappa-1)}(\delta-1, \gamma) + \delta G_{l+1}^{(\kappa)}(\delta-1, \gamma) \right)$$

$$\begin{aligned}
&= \frac{1}{\gamma^2(l+1)(l+2)} \left(\kappa(\kappa-1) G_{l+2}^{(\kappa-2)}(\delta-2, \gamma) + \kappa(\delta-1+\delta) G_{l+2}^{(\kappa-1)}(\delta-2, \gamma) \right. \\
&\quad \left. + \delta(\delta-1) G_{l+2}^{(\kappa)}(\delta-2, \gamma) \right) \\
&\vdots \\
&= (-1)^\delta \frac{\delta!}{\gamma^\delta(l+1)\dots(l+\delta)} \sum_{\omega=0}^{\delta} \binom{\delta}{\omega} \omega! H\langle \delta, \omega \rangle G_{l+\delta}^{(\kappa-\omega)}(0, \gamma) \\
&= (-1)^{\delta+\kappa} \frac{\delta! \kappa!}{\gamma^\delta(l+1)\dots(l+\delta)} \sum_{\omega=0}^{\delta} (-1)^\omega H\langle \delta, \omega \rangle \sum_{\wp=0}^{l+\delta} \binom{l+\delta}{\wp} \frac{\gamma^\wp}{(\wp+1)^{\kappa-\omega+1}},
\end{aligned}$$

as claimed. $\square$

For example, when $\gamma = \alpha$, $\delta = 1$ and $\kappa = 0$, by (1.2) and Binomial Theorem,

$$\begin{aligned}
\sum_{\wp=0}^l \binom{l}{\wp} \frac{\alpha^{\wp+1}}{\wp+2} &= \binom{l+1}{l}^{-1} \sum_{i=0}^1 \sum_{\wp=0}^{l+1} \binom{l+1}{\wp} \frac{(-1)^{i+1} \alpha^\wp}{(\wp+1)^{1-i}} H\langle 1, i \rangle \\
&= \frac{1}{l+1} \left(\sum_{\wp=0}^{l+1} \binom{l+1}{\wp} \alpha^\wp H\langle 1, 1 \rangle - \sum_{\wp=0}^{l+1} \binom{l+1}{\wp} \frac{\alpha^\wp}{\wp+1} H\langle 1, 0 \rangle \right) \\
&= \frac{1}{l+1} \left((1+\alpha)^{l+1} - \sum_{\wp=0}^{l+1} \binom{l+1}{\wp} \frac{\alpha^\wp}{\wp+1} \right),
\end{aligned}$$

and from here,

$$\sum_{\wp=1}^l \binom{l}{\wp} \alpha^\wp = \alpha^{2l} - 1. \quad (2.6)$$

Similarly, when $\gamma = \beta$, $\delta = 1$ and $\kappa = 0$,

$$\sum_{\wp=1}^l \binom{l}{\wp} \beta^\wp = \beta^{2l} - 1. \quad (2.7)$$

Combining (2.6) and (2.7), by Binet formula of Fibonacci numbers, we have

$$\sum_{\wp=0}^l \binom{l}{\wp} F_\wp = F_{2l} \text{ [24]}.$$

Corollary 2.4. For nonnegative integer l and positive integer δ , it holds that

$$\begin{aligned}
\sum_{\omega=0}^{\delta} \sum_{\wp=0}^{l+\delta} (-1)^{\omega+\wp} \binom{l+\delta}{\wp} (\wp+1)^{\omega-2} H\langle \delta, \omega \rangle &= \frac{\binom{l+\delta}{l}}{\binom{\delta+l+1}{l}(\delta+1)} H_{l,1}(\delta, 1), \\
\sum_{\omega=0}^{\delta} \sum_{\wp=0}^{l+\delta} (-1)^{\omega+\wp} \binom{l+\delta}{\wp} (\wp+1)^{\omega-3} H\langle \delta, \omega \rangle &= \binom{l+\delta}{l} \frac{H_{l,1}^2(\delta, 1) + H_{l,2}(\delta, 1)}{2! \binom{\delta+l+1}{l}(\delta+1)},
\end{aligned}$$

and

$$\sum_{\omega=0}^{\delta} \sum_{\wp=0}^{l+\delta} (-1)^{\omega+\wp} \binom{l+\delta}{\wp} (\wp+1)^{\omega-4} H\langle \delta, \omega \rangle = \binom{l+\delta}{l} \frac{2H_{l,3}(\delta, 1) + 3H_{l,1}(\delta, 1)H_{l,2}(\delta, 1) + H_{l,1}^3(\delta, 1)}{3! \binom{\delta+l+1}{l}(\delta+1)}.$$

Proof. Taking $\kappa = 1$ and $\gamma = -1$ in (2.5), we have

$$\binom{l+\delta}{l} \sum_{\wp=0}^l \binom{l}{\wp} \frac{(-1)^\wp}{(\wp+\delta+1)^2} = \sum_{\omega=0}^{\delta} \sum_{\wp=0}^{l+\delta} (-1)^\omega H_{\delta,\omega}(1) \binom{l+\delta}{\wp} \frac{(-1)^\wp}{(\wp+1)^{2-\omega}},$$

and from (1.3), the first result is given. Similarly, for $\kappa = 2, 3$ and $\gamma = -1$, the other results are obtained. Hence, the evidence is conclusive. $\square$

Lemma 2.5. For nonnegative integer n and positive integer κ , it holds that

$$G_l^{(\kappa)}(\delta, \gamma) = \int_0^1 (1 + \gamma t)^l (\log t)^\kappa t^\delta dt = (-1)^\kappa \kappa! \sum_{\wp=0}^l \binom{l}{\wp} \frac{\gamma^\wp}{(\wp + \delta + 1)^{\kappa+1}}, \quad (2.8)$$

where δ and γ are as above.

Proof. From Binomial Theorem and the integration by parts, the result is clearly given. $\square$

It is clearly seen that

$$G_{l+1}^{(\kappa)}(\delta, \gamma) = G_l^{(\kappa)}(\delta, \gamma) + \gamma G_l^{(\kappa)}(\delta + 1, \gamma).$$

Theorem 2.6. For positive integer κ , it holds that

$$\sum_{h=1}^{\infty} \frac{(-1)^h}{h} G_h^{(\kappa)}(0, \gamma) = (-1)^\kappa \kappa! (\kappa - \log \gamma),$$

where γ is positive real number.

Proof. Consider that

$$\begin{aligned} \sum_{h=1}^{\infty} \frac{(-1)^h}{h} \sum_{\wp=0}^h \binom{h}{\wp} \frac{\gamma^\wp}{(\wp + 1)^{\kappa+1}} &= \sum_{h=1}^{\infty} \frac{(-1)^h}{h} \int_0^1 \int_0^1 \dots \int_0^1 (1 + \gamma (x_1 x_2 \dots x_k))^h dx_1 dx_2 \dots dx_k \\ &= \int_0^1 \int_0^1 \dots \int_0^1 \sum_{h=1}^{\infty} \frac{(-1)^h}{h} (1 + \gamma (x_1 x_2 \dots x_k))^h dx_1 dx_2 \dots dx_k \\ &= - \int_0^1 \int_0^1 \dots \int_0^1 \log (\gamma x_1 x_2 \dots x_k) dx_1 dx_2 \dots dx_k \\ &= - \int_0^1 \int_0^1 \dots \int_0^1 (\log x_1 + \log x_2 + \dots + \log x_k + \log \gamma) \cdot dx_1 dx_2 \dots dx_k. \end{aligned}$$

From definition of improper integrals, we have

$$\begin{aligned} &\sum_{h=1}^{\infty} \frac{(-1)^h}{h} \sum_{\wp=0}^h \binom{h}{\wp} \frac{\gamma^\wp}{(\wp + 1)^{\kappa+1}} \\ &= - \int_0^1 \dots \int_0^1 \lim_{s \rightarrow 0^+} \int_s^1 (\log x_1 + \log x_2 + \dots + \log x_k + \log \gamma) dx_1 dx_2 \dots dx_k \\ &= - \int_0^1 \dots \int_{x_2=0}^1 (-1 + \log x_2 + \dots + \log x_k + \log \gamma) dx_2 \dots dx_k \\ &\vdots \\ &= - \int_{x_k=0}^1 (1 - \kappa + \log x_k + \log \gamma) dx_k \\ &= - \lim_{t \rightarrow 0^+} \int_{x_k=t}^1 (1 - \kappa + \log x_k + \log \gamma) dx_k \\ &= - \lim_{t \rightarrow 0^+} (1 - \kappa + \log \gamma - 1 - (1 - \kappa + \log \gamma) t + t \log t - t) \\ &= \kappa - \log \gamma, \end{aligned}$$

as claimed. $\square$

Lemma 2.7. For positive integer κ , it holds that

$$\sum_{h=0}^{\infty} H_{h,\kappa}(\delta, \gamma) t^h = \frac{1}{1-t} \Phi_{\kappa}(\gamma t, \delta+1), \quad (2.9)$$

where δ and γ are as above.

Proof. Consider that

$$\begin{aligned} (1-t) \sum_{h=0}^{\infty} H_{h,\kappa}(\delta, \gamma) t^h &= \sum_{h=0}^{\infty} H_{h,\kappa}(\delta, \gamma) t^h - \sum_{h=1}^{\infty} H_{h-1,\kappa}(\delta, \gamma) t^h \\ &= \frac{1}{(\delta+1)^{\kappa}} + \sum_{h=1}^{\infty} (H_{h,\kappa}(\delta, \gamma) - H_{h-1,\kappa}(\delta, \gamma)) t^h \\ &= \frac{1}{(\delta+1)^{\kappa}} + \sum_{h=1}^{\infty} \frac{\gamma^h t^h}{(h+\delta+1)^{\kappa}} \\ &= \sum_{h=0}^{\infty} \frac{\gamma^h t^h}{(h+\delta+1)^{\kappa}} = \Phi_{\kappa}(\gamma t, \delta+1), \end{aligned}$$

as claimed. $\square$

Theorem 2.8. For nonnegative integer l and positive integer κ , it holds that

$$\sum_{\wp=0}^l \left(\frac{l-\wp+1}{(\wp+\delta+1)^{\kappa}} + \frac{(\wp+1)\gamma}{(\wp+\delta+2)^{\kappa}} \right) \gamma^{\wp} = (l+1) H_{l+1,\kappa}(\delta, \gamma), \quad (2.10)$$

where δ and γ are as above.

Proof. By (2.9), we get

$$\begin{aligned} \sum_{h=0}^{\infty} (h+1) H_{h+1,\kappa}(\delta, \gamma) t^h &= \frac{1}{(1-t)^2} \Phi_{\kappa}(\gamma t, \delta+1) + \frac{1}{1-t} \sum_{h=1}^{\infty} \frac{h \gamma^h t^{h-1}}{(h+\delta+1)^{\kappa}} \\ &= \sum_{h=0}^{\infty} (h+1) t^h \sum_{h=0}^{\infty} \frac{\gamma^h t^h}{(h+\delta+1)^{\kappa}} + \sum_{h=0}^{\infty} t^h \sum_{h=0}^{\infty} \frac{(h+1) \gamma^{h+1} t^h}{(h+\delta+2)^{\kappa}} \\ &= \sum_{h=0}^{\infty} \sum_{\wp=0}^h (h-\wp+1) \frac{\gamma^{\wp} t^h}{(\wp+\delta+1)^{\kappa}} + \sum_{h=0}^{\infty} \sum_{\wp=0}^h \frac{(\wp+1) \gamma^{\wp+1} t^h}{(\wp+\delta+2)^{\kappa}} \\ &= \sum_{h=0}^{\infty} \sum_{\wp=0}^h \left(\frac{h-\wp+1}{(\wp+\delta+1)^{\kappa}} + \frac{(\wp+1)\gamma}{(\wp+\delta+2)^{\kappa}} \right) \gamma^{\wp} t^h. \end{aligned}$$

So, through the comparison of coefficients on each side, the demonstration is concluded. $\square$

Theorem 2.9. For nonnegative integer l and positive integer κ , it holds that

$$H_{l,\kappa}(\delta, \gamma) = \sum_{j=0}^l \sum_{\wp=0}^j \sum_{\omega=0}^{\wp} (-1)^{\omega} \binom{j}{\wp} \frac{\gamma^{l-j} \wp!}{\omega! j! (l-j+\delta+1)^{\kappa}}, \quad (2.11)$$

and

$$H_{l,\kappa}(\delta, \gamma) = \sum_{j=0}^l \sum_{\wp=0}^j (-1)^{\wp} \frac{\gamma^{l-j} C_{\wp} H_{j-\wp+1}}{\wp! (l-j+\delta+1)^{\kappa}}, \quad (2.12)$$

where δ and γ are as above.

Proof. From (1.5) and (2.9), we have

$$\begin{aligned} \sum_{h=0}^{\infty} H_{h,\kappa}(\delta, \gamma) t^h &= e^t e^{-t} \frac{1}{1-t} \Phi_{\kappa}(\gamma t, \delta+1) \\ &= \sum_{h=0}^{\infty} \frac{t^h}{h!} \sum_{h=0}^{\infty} (-1)^h \frac{t^h}{h!} \sum_{h=0}^{\infty} t^h \sum_{h=0}^{\infty} \frac{\gamma^h}{(h+\delta+1)^{\kappa}} t^h \\ &= \sum_{h=0}^{\infty} \frac{t^h}{h!} \sum_{h=0}^{\infty} \sum_{\omega=0}^h (-1)^{\omega} \frac{t^h}{\omega!} \sum_{h=0}^{\infty} \frac{\gamma^h}{(h+\delta+1)^{\kappa}} t^h \\ &= \sum_{h=0}^{\infty} \sum_{\wp=0}^h \sum_{\omega=0}^{\wp} \frac{(-1)^{\omega}}{\omega! (h-\wp)!} t^h \sum_{h=0}^{\infty} \frac{\gamma^h}{(h+\delta+1)^{\kappa}} t^h \\ &= \sum_{h=0}^{\infty} \sum_{j=0}^h \sum_{\wp=0}^j \sum_{\omega=0}^{\wp} \frac{(-1)^{\omega}}{\omega! (j-\wp)!} \frac{\gamma^{h-j}}{(h-j+\delta+1)^{\kappa}} t^h. \end{aligned}$$

Thus, by comparing the coefficients on both sides, (2.11) can be seen. Also, with the help of (1.1), (1.4) and (2.9), the proof of (2.12) is similar to the proof of (2.11). $\square$

Theorem 2.10. For nonnegative integer l and positive integer κ , it holds that

$$\begin{aligned} \sum_{\wp=0}^l \frac{(-1)^{\wp} \wp!}{(\wp+\delta+1)^{\kappa}} \gamma^{\wp} S_2(l, \wp) &= \sum_{\omega=0}^l \sum_{\wp=0}^{\omega} (-1)^{\wp} \binom{l}{\omega} \wp! H_{\wp,\kappa}(\delta, \gamma) S_2(\omega, \wp), \\ \sum_{\wp=0}^l (-1)^{\wp} \wp! H_{\wp,\kappa}(\delta, \gamma) S_2(l, \wp) &= \sum_{\omega=0}^l \sum_{\wp=0}^{\omega} \binom{l}{\omega} \frac{(-1)^{\wp+\eta-\omega} \wp! \gamma^{\wp}}{(\wp+\delta+1)^{\kappa}} S_2(\omega, \wp), \end{aligned}$$

and

$$\sum_{\wp=0}^l (-1)^{\wp} (\wp+1)! S_2(l, \wp) H_{\wp+1,\kappa}(\delta, \gamma) = \sum_{\wp=0}^l \sum_{\omega=0}^{\wp} (-1)^{\wp} \wp! \left(\frac{\wp-\omega+1}{(\omega+\delta+1)^{\kappa}} + \frac{(\omega+1)\gamma}{(\omega+\delta+2)^{\kappa}} \right) \gamma^{\omega} S_2(l, \wp),$$

where δ and γ are as above.

Proof. By taking $t \rightarrow 1 - e^t$ in (2.9), it is easily seen that

$$e^t \sum_{l=0}^{\infty} l! H_{l,\kappa}(\delta, \gamma) \frac{(1-e^t)^l}{l!} = \sum_{\wp=0}^{\infty} \frac{\wp! \gamma^{\wp}}{(\wp+\delta+1)^{\kappa}} \frac{(1-e^t)^{\wp}}{\wp!}.$$

From here, (1.5) and (1.6) yield that

$$\begin{aligned} e^t \sum_{\wp=0}^{\infty} (-1)^{\wp} \wp! H_{\wp,\kappa}(\delta, \gamma) \frac{(e^t-1)^{\wp}}{\wp!} &= e^t \sum_{\wp=0}^{\infty} (-1)^{\wp} \wp! H_{\wp,\kappa}(\delta, \gamma) \sum_{l=\wp}^{\infty} S_2(l, \wp) \frac{t^l}{l!} \\ &= \sum_{l=0}^{\infty} \frac{t^l}{l!} \sum_{l=0}^{\infty} \sum_{\wp=0}^l (-1)^{\wp} \wp! H_{\wp,\kappa}(\delta, \gamma) S_2(l, \wp) \frac{t^l}{l!} \\ &= \sum_{l=0}^{\infty} \sum_{\omega=0}^l \sum_{\wp=0}^{\omega} (-1)^{\wp} \wp! H_{\wp,\kappa}(\delta, \gamma) S_2(\omega, \wp) \binom{l}{\omega} \frac{t^l}{l!}, \end{aligned}$$

and

$$\sum_{\wp=0}^{\infty} \frac{(-1)^{\wp} \wp! \gamma^{\wp}}{(\wp+\delta+1)^{\kappa}} \frac{(e^t-1)^{\wp}}{\wp!} = \sum_{\wp=0}^{\infty} \frac{(-1)^{\wp} \wp! \gamma^{\wp}}{(\wp+\delta+1)^{\kappa}} \sum_{l=\wp}^{\infty} S_2(l, \wp) \frac{t^l}{l!}.$$

Thus, by combining the coefficients on both sides, the result is obtained. Taking $t \rightarrow 1 - e^t$ in (2.10), the other results are similarly obtained. $\square$

Some consequences of Theorem 2.10 will now be stated as examples. Using Definition 2.1, then

$$\sum_{\wp=0}^l \frac{(-1)^\wp \wp!}{(\wp + \delta + 1)^\kappa} \gamma^\wp S_2(l, \wp) = \sum_{i=0}^l \sum_{\wp=0}^i \sum_{t=0}^\wp \binom{l}{i} \frac{(-1)^\wp \wp!}{(t + \delta + 1)^\kappa} \gamma^t S_2(i, \wp).$$

Here, taking α, β replace of γ , respectively and $\delta = 0$,

$$\sum_{\wp=0}^l \frac{(-1)^\wp \wp!}{(\wp + 1)^\kappa} F_\wp S_2(l, \wp) = \sum_{i=0}^l \sum_{\wp=0}^i \sum_{t=0}^\wp \binom{l}{i} \frac{(-1)^\wp \wp!}{(t + 1)^\kappa} F_t S_2(i, \wp).$$

Similarly, it is clearly seen that

$$\sum_{\wp=0}^l \sum_{t=0}^\wp \frac{(-1)^\wp \wp!}{(t + 1)^\kappa} F_t S_2(l, \wp) = \sum_{i=0}^l \sum_{\wp=0}^i \binom{l}{i} \frac{(-1)^{\wp+l-i} \wp!}{(\wp + 1)^\kappa} F_\wp S_2(i, \wp),$$

and

$$\sum_{\wp=0}^l \sum_{t=0}^{\wp+1} \frac{(-1)^\wp (\wp + 1)!}{(t + 1)^\kappa} F_t S_2(l, \wp) = \sum_{\wp=0}^l \sum_{i=0}^\wp (-1)^\wp \wp! \left(\frac{\wp - i + 1}{(i + 1)^\kappa} + \frac{(i + 1)}{(i + 2)^\kappa} \right) F_i S_2(l, \wp).$$

Theorem 2.11. For nonnegative integer l , it holds that

$$\sum_{\wp=0}^l \sum_{\omega=0}^{l-\wp} \frac{1}{\omega + \wp + 1} \binom{l-\wp}{\omega} \left(\binom{\delta+l+1}{\wp} \gamma^\omega - \binom{\delta+\wp}{\wp} (\gamma+1)^\omega \right) = 0, \quad (2.13)$$

where δ and $\gamma \setminus \{0, -1\}$ are as above.

Proof. From the Vandermonde convolution $\sum_{\omega=0}^l \binom{\delta+\omega}{\omega} \binom{\eta-\omega}{\ell-\omega} = \binom{\delta+\eta+1}{\ell}$, we write

$$\begin{aligned} \sum_{\wp=0}^l \binom{\delta+l+1}{\wp} t^\wp (1+\gamma t)^{l-\wp} &= \sum_{\wp=0}^l \sum_{\omega=0}^\wp \binom{\delta+\omega}{\omega} \binom{l-\omega}{\wp-\omega} t^\wp (1+\gamma t)^{l-\wp} \\ &= \sum_{\omega=0}^l \binom{\delta+\omega}{\omega} \sum_{\wp=\omega}^l \binom{l-\omega}{\wp-\omega} t^\wp (1+\gamma t)^{l-\wp} \\ &= \sum_{\omega=0}^l \binom{\delta+\omega}{\omega} t^\omega \sum_{\wp=0}^{l-\omega} \binom{l-\omega}{\wp} t^\wp (1+\gamma t)^{l-\omega-\wp} \\ &= \sum_{\omega=0}^l \binom{\delta+\omega}{\omega} t^\omega (1+(\gamma+1)t)^{l-\omega}. \end{aligned} \quad (2.14)$$

From here, by (2.8), the proof follows from integrating both sides of (2.14) from $t = 0$ to $t = 1$. $\square$

For example, taking $\delta = 0$ and $\gamma = 1$ in (2.13), it is seen that

$$\sum_{\wp=0}^l \sum_{\omega=0}^{l-\wp} \frac{1}{\omega + \wp + 1} \binom{l-\wp}{\omega} \left(\binom{l+1}{\wp} - 2^\omega \right) = 0.$$

3. CONCLUSIONS

In the current paper, the study aim to obtain integral and generating function with the help of the generalized harmonic numbers and give some identities. For that we define the generalized harmonic numbers with parameters δ and γ , $H_{l,\kappa}(\delta, \gamma)$. Then, we obtain $G_l^{(\kappa)}(\delta, \gamma) = \int_0^1 (1+\gamma t)^l (\log t)^\kappa t^\delta dt$ and give the recurrence relations of them. Also, we obtain the value to which the infinite series $\sum_{h=1}^{\infty} \frac{(-1)^h}{h} G_h^{(\kappa)}(0, \gamma)$ converges. Considering generating function of $H_{l,\kappa}(\delta, \gamma)$, we investigate some identities involving Stirling numbers and Fibonacci numbers. Researchers can continue to improve the work by differentiating m times with respect to γ on $G_l(\delta, \gamma) = \sum_{\omega=0}^l \binom{l}{\omega} \frac{\gamma^\omega}{\omega + \delta + 1}$. The graphs given in

the complex plane are 2D and 3D visualizations showing the root distributions, and reveal how the gamma and delta parameters form a structure in their real and imaginary components. The curvilinear patterns of the points show that the roots are clustered in a certain order and provide important information about the parametric stability of the system or the distribution points. Future works can deepen these results with different parameter ranges or higher-dimensional spectral analyses and examine the dynamic behavior of the system more comprehensively. In addition, work can be conducted using analytical and numerical methods together to strengthen the relationship between root distributions and physical or engineering problems.

AUTHORS CONTRIBUTION STATEMENT

All authors contributed equally. All authors have read and approved the final manuscript.

CONFLICTS OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this article.

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