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Exploring Harmonic and Hyper-harmonic Narayana Numbers through Circulant Structures and Matrix Norms

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Highlights

- Harmonic and hyper-harmonic Narayana numbers yield new identities and closed forms.
- Narayana-type circulant and r-circulant matrices are studied via matrix norms.
- Inequalities for matrix norms, linking special number sequences with structured matrix analysis.

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Abstract

This study investigates the structural and analytical properties of harmonic and hyper-harmonic Narayana numbers, extending classical results on harmonic and hyper-harmonic numbers. We establish novel combinatorial identities and derive closed-form expressions for $\aleph_n$ and $\aleph_n^{(r)}$, which represent finite sums involving reciprocals of Narayana numbers. Furthermore, we examine the spectral and Euclidean norms of circulant and r-circulant matrices generated by these numbers, together with selected special matrix forms. Through this framework, several inequalities associated with matrix norms are obtained, providing deeper insight into the interplay between Narayana-type sequences and matrix analysis. The results contribute both to the combinatorial theory of special number sequences and to the normative analysis of structured matrices.

1. INTRODUCTION

The Narayana numbers, introduced by the 14th-century Indian mathematician Narayana, arise from a classical population growth problem with delayed reproduction [1]. Unlike the Fibonacci sequence, they satisfy a third-order recurrence relation

$$N_{n+1} = N_n + N_{n-2}, N_0 = 0, N_1 = 1, N_2 = 1$$

generating the sequence 0,1,1,1,2,3,4,6,9,13,… [2,3]. This structure leads to rich combinatorial and algebraic properties, often studied via generating functions and matrix representations. Although the sequence is historically derived from the "Narayana's cows" problem, it has become standard in contemporary mathematical literature to refer to it as the Narayana numbers. This terminology has been adopted in numerous studies dealing with combinatorial, algebraic, and matrix-theoretic aspects of the sequence. Therefore, in line with the existing literature, we use the term *Narayana numbers* throughout this paper.

Recent works have extended Narayana numbers to k-generalizations, polynomials, and hybrid or complex forms, revealing connections with Pascal-type triangles, self-similar graph structures, and generalized number systems [4,5]. Motivated by these developments, this study introduces structural and analytical properties of harmonic and hyper-harmonic Narayana numbers and investigates its fundamental properties.

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In analogy with harmonic numbers defined via reciprocals of integers, it is natural to consider similar constructions based on Narayana numbers, leading to the concept of harmonic Narayana numbers.

Every Harmonic numbers boast a long and fascinating history, intertwining with the concept of the harmonic mean. The nth harmonic number can be envisioned as n multiplied by the reciprocal of the average obtained when we take the reciprocals of the first n positive integers (1 divided by each integer).

These captivating numbers hold significant value across various branches of number theory. Often referred to informally as the harmonic series, they exhibit a close connection with the Riemann zeta function and frequently appear within expressions for specialized functions.

Interestingly, harmonic numbers offer an approximation to the natural logarithm function [6]. As a consequence, the corresponding harmonic series diverges, meaning its sum grows infinitely large, albeit at a slow rate. In 1737, the brilliant mathematician Leonhard Euler ingeniously leveraged this divergence to establish a novel proof for the infinitude of prime numbers. Building upon Euler's groundbreaking work, Bernhard Riemann extended the concept into the complex plane in 1859. This work paved the way for the Riemann hypothesis, a cornerstone in comprehending the distribution of prime numbers. Furthermore, when dealing with vast quantities of items that follow a Zipf's law distribution (where the value of an item is inversely proportional to its rank), the total value of the n most valuable items demonstrates proportionality to the nth harmonic number. This intriguing relationship unlocks unexpected insights into the concept of the long tail (the presence of many low-value items) and the theory of network value (the significance of connections within a network).

In mathematics, the n-th harmonic number serves as a way to calculate a specific value by summing the reciprocals (1 divided by the number) of the first n positive integers (whole numbers starting from 1). The mth harmonic number is defined by

$$H_m = \sum_{k=1}^{m} \frac{1}{k}$$

where $H_0 = 0$. The mth harmonic number can be expressed in another way as follows:

$$H_m = \frac{\left[\begin{smallmatrix} m+1 \\ 2 \end{smallmatrix}\right]}{m!}$$

In this context, $\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]$ represents the Stirling number of the first kind, written as $[m, k]$. It counts the number of variations you can create by arranging m distinct elements into k separate, non-overlapping circles (cycles). Also, some classical combinatorial identities involving binomial coefficients will be used in the sequel. In particular, we recall the identity

$$\sum_{k=0}^{n} \binom{k+r-1}{r-1} = \binom{n+r}{r}.$$

which plays a fundamental role in the derivation of summation formulas related to hyper-harmonic-type structures [7].

Ohtsuka and Nakamura [8] examined the partial infinite sums of the reciprocals of Fibonacci numbers and their squares, and Holiday and Komatsu [9] subsequently generalized these results. Inspired by these studies Tuğlu et al. [10] and Çetin et al. [11] study harmonic and hyper-harmonic Fibonacci numbers and their combinatorial properties. Dişkaya and Menken [12] also approached the study from a different dimension working with Padovan numbers having three recurrences.

Harmonic numbers have been extensively studied by numerous mathematicians, who have explored their diverse facets. Recent studies on harmonic and hyper-harmonic numbers have revealed rich combinatorial structures and deep connections with generating functions and special sequences. In particular, various identities involving harmonic and hyper-harmonic numbers have been derived using umbral calculus and associated sequence techniques [13]. Generalized harmonic and hyper-harmonic numbers have also been extensively studied through generating functions and combinatorial sums involving Stirling, Daehee, and related special numbers [14]. Furthermore, relationships between hyper-harmonic numbers and Daehee numbers have been established via differential equations and generating function approaches [15]. From a matrix-theoretic perspective, harmonic-type sequences have been incorporated into structured matrices, leading to results on determinants, inverses, and norms, particularly for matrices with entries involving generalized Fibonacci-type numbers [16]. In addition, harmonic and hyper-harmonic extensions of higher-order Gauss Fibonacci sequences have been introduced, together with their generating functions and algebraic properties [17]. These contributions collectively demonstrate the versatility of harmonic-type constructions and motivate their extension to other special number sequences.

The study of Circulant matrices dates back to 1956, with Ingleton's [18] pioneering work on their rank operations. Subsequently, Philip [19] made significant contributions to the field. In the late 90s, Golub et al. [20] focused on computational aspects of Circulant systems. More recently, Kumari et al. [21] initiated research linking Circulant matrices with some number sequences, followed by Kuloğlu et al. [22] who expanded these studies to k-Fermat and k-Mersenne numbers, culminating in broader investigations encompassing more general forms of Fermat and Mersenne numbers [23]. These works marked the conclusion of focused research on Circulant matrices within this specific number series.

However, the exploration of Circulant matrices continued in other avenues. Falcon [24] studied them in the context of k-Fibonacci matrices, while Uygun et al. [25] delved into Jacobsthal and Jacobsthal Lucas numbers. Additionally, researchers like Bozkurt et al. [26], Bueno [27], Solak [28], and Zheng et al. [29] investigated various special number sequences like Pell, Jacobsthal, Fibonacci, and Lucas in relation to Circulant matrices. Further studies explored bounds for the norms of r-circulant matrices, as seen in the work of Shen et al. [30].

Also, Tuğlu and Kızılateş studied circulant and r-circulant matrices on matrices whose entries are harmonic and hyper-harmonic Fibonacci numbers and studied their norms and some other properties [31,32].

Definition 1.1. The circulant matrix of any K matrix is an $n \times n$ matrix and is defined as follows [33]

$$K = \begin{pmatrix} k_0 & k_1 & k_2 & \cdots & k_{n-2} & k_{n-1} \\ k_{n-1} & k_0 & k_1 & \cdots & k_{n-3} & k_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k_1 & k_2 & k_3 & \cdots & k_{n-1} & k_0 \end{pmatrix}.$$

Here, it can be seen that the elements of the matrix K are $\{k_0, k_1, k_2, \dots, k_{n-2}, k_{n-1}\}$ given in the first row and this matrix can be essentially defined as $K = Circ(k_0, k_1, k_2, \dots, k_{n-2}, k_{n-1})$. Similarly, the first study of r-circulant matrices was done by mathematician Douglas Lind in the 1970s. Lind studied circulant matrices defined by Fibonacci numbers and discovered some properties of these matrices. Later, investigations have yielded a plethora of identities, properties, theorems, and generalizations pertaining to norm of circulant and r-circulant matrices [34-40].

The r-circulant matrix of matrix K is an $n \times n$ matrix and is defined as follows:

$$K_r = \begin{pmatrix} k_0 & k_1 & k_2 & \cdots & k_{n-2} & k_{n-1} \\ rk_{n-1} & k_0 & k_1 & \cdots & k_{n-3} & k_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rk_1 & rk_2 & rk_3 & \cdots & rk_{n-1} & k_0 \end{pmatrix}.$$

Let $S = (s_{ij})$ be a matrix of type $a \times b$. The Euclidean norm is the length of the vector formed by stacking the matrix columns. The spectral norm is the largest singular value of the matrix. So that:

$$\|S\|_E = \sqrt{\sum_{i=1}^{a}\sum_{j=1}^{b}|s_{ij}|^2} \tag{1}$$

and

$$\|S\|_2 = \sqrt{\underbrace{max}_{1\le i\le n}\ \delta_i(S^H S)}\,. \tag{2}$$

Equation (2) is the largest singular value of the matrix.

Here $\delta_i(S^H S)$, is the eigenvalues of $S^H S$ and S^H is the conjugate transpose of the matrix S.
Then it can be said that there is a relationship so that, the spectral norm still provides an upper bound for the Euclidean norm, offering valuable information about the matrix's influence on vector lengths:

$$\frac{1}{\sqrt{n}}\|S\|_E \le \|S\|_2 \le \|S\|_E. \tag{3}$$

Let us consider two different matrices M and N of type $a \times b$. Then we obtain the following inequalities from [29]:

$$\|M \circ N\|_2 \le \|M\|_2\|N\|_2. \tag{4}$$

In Equation (5), $M \circ N$ is Hadamard products of the matrices M and N. Also,

$$\|M \circ N\|_2 \le r_1(M)r_2(N) \tag{5}$$

where

$$r_1(M) = \underbrace{max}_{1\le i\le a}\sqrt{\sum_{j=1}^{b}|m_{ij}|^2}$$

and

$$r_2(N) = \underbrace{max}_{1\le j\le b}\sqrt{\sum_{i=1}^{a}|n_{ij}|^2}.$$

In [41], finite difference operator defined as $\Delta\mathfrak{f}(x) = \mathfrak{f}(x+1) - \mathfrak{f}(x)$.
Finite difference operator has an operator defined as an anti-difference operator and denoted by Σ. Let $m\ and\ n$ are integers with $n \ge m$. If $\mathfrak{g}(x) = \Delta\mathfrak{f}(x)$ then

$$\sum_{m}^{n}\mathfrak{g}(x)\lambda_x = \sum_{x=m}^{n-1}\mathfrak{g}(x) = \mathfrak{f}(n) - \mathfrak{f}(m). \tag{6}$$

In the following, some basic properties of the anti-difference operator are analysed:

$$\sum_{m}^{n} s(x)\Delta t(x)\,\lambda_x = s(x)t(x)|_{m}^{n+1} - \sum_{m}^{n} t(x+1)\Delta s(x)\lambda_x \tag{7}$$

and for $u \neq -1$,

$$\sum x^{\underline{u}}\lambda_x = \frac{x^{\underline{u+1}}}{u+1},$$

where $x^{\underline{u}} = x(x-1)(x-2)\ldots(x-u+1)$.

2. HARMONIC, HYPER-HARMONIC NARAYANA NUMBERS AND SOME COMBINATORIAL PROPERTIES

In this section harmonic, hyper-harmonic Narayana numbers are defined and some properties, sums and related theorems and propositions are discussed.

Definition 2.1. For positive integer r, hyper-Narayana numbers are defined by

$$N_n^{(r)} = \sum_{k=0}^{n} N_k^{(r-1)}$$

with $N_n^{(0)} = N_n, N_0^{(r)} = 0, N_1^{(r)} = 1$.

Definition 2.2. Let N_k, kth Narayana number. The nth harmonic number $\aleph_n$is defined by

$$\aleph_n = \sum_{k=1}^{n} \frac{1}{N_k}, n \geq 1. \tag{8}$$

Below are the first few harmonic Narayana numbers:

$$\aleph_1 = 1, \aleph_2 = 2, \aleph_3 = 3, \aleph_4 = \frac{7}{2}, \aleph_5 = \frac{23}{6}, \aleph_6 = \frac{49}{12}, \aleph_7 = \frac{17}{4}, \aleph_8 = \frac{157}{36}$$

Theorem 2.3. Let $\aleph_n$, nth harmonic Narayana number. Then we have

$$\sum_{k=0}^{n-1} \aleph_k = n\aleph_n - \sum_{k=0}^{n-1} \frac{k+1}{N_{k+1}}. \tag{9}$$

Proof. According to the equation given in Equation (7), let $t(k) = \aleph_k$ and $\Delta s(k) = 1$. So that we have $\Delta t(k) = \frac{1}{N_{k+1}}$, $s(k) = k$ and $s(k+1) = k+1$.

From here we get the equation given below:

$$\sum_{k=0}^{n-1} \aleph_k \varphi_k = k\aleph_k|_0^n - \sum_{k=0}^{n-1} \frac{k+1}{N_{k+1}} \varphi_k.$$

From Equation (6), we get the desired result.

Theorem 2.4. Let $\aleph_n$ be nth harmonic Narayana number. Then we have

$$\sum_{k=0}^{n-1}(\aleph_k)^2 = n\,(\aleph_n)^2 - \sum_{k=0}^{n-1}\frac{k+1}{N_{k+1}}\left(2\aleph_k + \frac{1}{N_{k+1}}\right). \tag{10}$$

Proof. If we take $t(k) = (\aleph_k)^2$ and $\Delta s(k) = 1$ then $s(k) = k, s(k+1) = k+1$.
Since $\Delta\aleph_k = \aleph_{k+1} - \aleph_k = \frac{1}{N_{k+1}}$, we obtain $\Delta t(k) = (\aleph_{k+1})^2 - (\aleph_k)^2$. Using $\aleph_{k+1} = \aleph_k + \frac{1}{N_{k+1}}$, we get

$$\Delta t(k) = \frac{1}{N_{k+1}}\left(2\aleph_k + \frac{1}{N_{k+1}}\right).$$

Then, if we substitute what is given in equation (7), we get

$$\sum_{k=0}^{n-1}(\aleph_k)^2\varphi_k = k\,(\aleph_k)^2|_0^n - \sum_{k=0}^{n-1}\frac{k+1}{N_{k+1}}\left(2\aleph_k + \frac{1}{N_{k+1}}\right)\varphi_k.$$

The desired result is obtained from Equation (6).

Theorem 2.5. Let $\aleph_n$ be nth harmonic Narayana number and $u \in \mathbb{Z}^+ \cup \{0\}$. Then we have

$$\sum_{k=0}^{n-1}\binom{k}{u}\aleph_k = \binom{n}{u+1}\aleph_n - \sum_{k=0}^{n-1}\binom{k+1}{u+1}\frac{1}{N_{k+1}}, n \geq 1. \tag{11}$$

Proof. Let $t(k) = \aleph_k, \Delta s(k) = \binom{k}{u}$. We have $\Delta t(k) = \frac{1}{N_{k+1}}, s(k) = \binom{k}{u+1}$. From (6) and (7), we get

$$\sum_{k=0}^{n-1}\binom{k}{u}\aleph_k\varphi_k = \binom{k}{u+1}\aleph_k|_0^n - \sum_{k=0}^{n-1}\binom{k+1}{u+1}\frac{1}{N_{k+1}}\varphi_k.$$

Since

$$\binom{k}{u+1}\aleph_k|_0^n = \binom{n}{u+1}\aleph_n,$$

we get

$$\sum_{k=0}^{n-1}\binom{k}{u}\aleph_k\varphi_k = \binom{n}{u+1}\aleph_n - \sum_{k=0}^{n-1}\binom{k+1}{u+1}\frac{1}{N_{k+1}}\varphi_k.$$

Theorem 2.6. Let $\aleph_n$ be nth harmonic Narayana number and $u \in \mathbb{Z}^+ \cup \{0\}$. Then we have:

$$\sum_{k=0}^{n-1}k^{\underline{u}}\aleph_k = \frac{n^{\underline{u+1}}}{u+1}\aleph_n - \sum_{k=0}^{n-1}\frac{(k+1)^{\underline{u+1}}}{u+1}.\frac{1}{N_{k+1}}, n \geq 1. \tag{12}$$

Proof. Let $t(k) = \aleph_k, \Delta s(k) = k^{\underline{u}}$. We have $\Delta t(k) = \frac{1}{N_{k+1}}, s(k) = \frac{(k)^{\underline{u+1}}}{u+1}$.

From Equation (7), we get desired Equation (12).

There are different relationships between harmonic numbers and harmonic Narayana numbers. We can examine these relationships with the help of the theorems given below.

Theorem 2.7. Let H_n, nth harmonic number and $\aleph_n$ be nth harmonic Narayana number. Then we get

$$\sum_{k=0}^{n-1} \frac{\aleph_k}{k+1} = H_n \aleph_n - \sum_{k=0}^{n-1} \frac{H_{k+1}}{N_{k+1}}. \tag{13}$$

Proof. Let $t(k) = \aleph_k,\ \Delta s(k) = \frac{1}{k+1}$. We have $\Delta t(k) = \frac{1}{N_{k+1}}, s(k) = H_k$. From Equation (6),

$$\sum_{k=0}^{n-1} \frac{\aleph_k}{k+1} \varphi_k = H_k \aleph_k|_0^n - \sum_{k=0}^{n-1} \frac{H_{k+1}}{N_{k+1}} \varphi_k.$$

The desired result is obtained with basic operations.

Corollary 2.8.

$$\sum_{k=0}^{n-1} \frac{\aleph_k}{k+1} = \frac{\begin{bmatrix} n+1 \\ 2 \end{bmatrix} \aleph_n}{n!} - \sum_{k=0}^{n-1} \frac{\begin{bmatrix} k+2 \\ 2 \end{bmatrix}}{(k+1)!\, N_{k+1}}.$$

Proof. If we take $H_n = \frac{\begin{bmatrix} n+1 \\ 2 \end{bmatrix}}{n!}$ in Theorem 2.7, the proof can be completed.

Theorem 2.9. Let $\aleph_n$ be nth harmonic Narayana number. Then we get

$$\sum_{k=0}^{n-1} N_{k-2} \aleph_k = N_n \aleph_n - n.$$

Proof. Let $t(k) = \aleph_k,\ \Delta s(k) = N_{k-2}$. We have $\Delta t(k) = \frac{1}{N_{k+1}}, s(k) = N_k$. From Equation (6), we get

$$\sum_{k=0}^{n-1} \aleph_k N_{k-2} \varphi_k = N_k \aleph_k|_0^n - \sum_{k=0}^{n-1} \frac{N_{k+1}}{N_{k+1}} \varphi_k$$

$$\sum_{k=0}^{n-1} N_{k-2} \aleph_k = N_n \aleph_n - n$$

as desired.

The definitions, theorems and propositions given below will give theorems that will help us examine it in more detail by enabling the formation of hyper-harmonic Narayana numbers with the help of the harmonic Narayana and hyper-Narayana numbers we created.

Definition 2.10. Let $\aleph_n$ be nth harmonic Narayana number. For $r, n \in \mathbb{Z}^+$, we have

$$\aleph_n^{(r)} = \sum_{k=1}^{n} \aleph_k^{(r-1)}, \aleph_n^{(0)} = \frac{1}{N_n}, \aleph_0^{(k)} = 0 \tag{14}$$

where $\aleph_n^{(r)}$, nth hyper-harmonic Narayana number.
In particular for $r = 1$, we get

$$\aleph_n^{(1)} = \sum_{k=1}^{n} \aleph_k^{(0)} = \sum_{k=1}^{n} \frac{1}{N_n} = \aleph_n.$$

From here it can be seen that $\aleph_n^{(1)}$=$\aleph_n$.

Below are the first few terms of the hyper-harmonic Narayana numbers according to some cases of n and r.

For $r = 1$, and $n = 1,2,3, \ldots$

$$\aleph_1^{(1)} = 1, \aleph_2^{(1)} = 2, \aleph_3^{(1)} = 3, \aleph_4^{(1)} = \frac{7}{2}, \aleph_5^{(1)} = \frac{23}{6}, \aleph_6^{(1)} = \frac{49}{12}, \ldots.$$

For $r = 2$, and $n = 1,2,3, \ldots$

$$\aleph_1^{(2)} = 1, \aleph_2^{(2)} = 3, \aleph_3^{(2)} = 6, \aleph_4^{(2)} = \frac{19}{2}, \aleph_5^{(2)} = \frac{40}{3}, \aleph_6^{(2)} = \frac{209}{12}, \ldots.$$

For $r = 3$, and $n = 1,2,3, \ldots$

$$\aleph_1^{(3)} = 1, \aleph_2^{(3)} = 4, \aleph_3^{(3)} = 10, \aleph_4^{(3)} = \frac{39}{2}, \aleph_5^{(3)} = \frac{197}{6}, \ldots.$$

Lemma 2.11. Hyper-harmonic Narayana number have a recurrence relation as follows:

$$\aleph_n^{(r)} = \aleph_{n-1}^{(r)} + \aleph_n^{(r-1)}. \tag{15}$$

Proof. From Equation (14), we have

$$\begin{aligned} \aleph_n^{(r)} &= \sum_{k=1}^{n} \aleph_k^{(r-1)} \\ &= \sum_{k=1}^{n-1} \aleph_k^{(r-1)} + \aleph_n^{(r-1)} \\ &= \aleph_{n-1}^{(r)} + \aleph_n^{(r-1)}. \end{aligned}$$

Theorem 2.12. For $1 \leq a, b \leq n$, we have

$$\aleph_{n-a+1}^{(b)} = \sum_{k=a}^{n} \binom{n-k+b-1}{b-1} \frac{1}{N_{k-a+1}}.$$

Proof. From the definition of hyper-harmonic Narayana numbers, we have

$$\aleph_n^{(b)} = \sum_{k_b=1}^{n} \aleph_{k_b}^{(b-1)}.$$

Applying the same definition repeatedly, we obtain

$$\aleph_n^{(b)} = \sum_{k_b=1}^{n} \sum_{k_{b-1}=1}^{k_b} .. \sum_{k_1=1}^{k_2} \frac{1}{N_{k_1}}.$$

Now, replacing n by $n - a + 1$, we write

$$\aleph_{n-a+1}^{(b)} = \sum_{k_b=1}^{n-a+1} \sum_{k_{b-1}=1}^{k_b} \ldots \sum_{k_1=1}^{k_2} \frac{1}{N_{k_1}}.$$

For a fixed index k, the term $\frac{1}{N_{k-a+1}}$ appears as many times as the number of ways to choose the remaining $b-1$ indices between k and n. This number is

$$\binom{n-k+b-1}{b-1}.$$

Now we continue the proof from here with the induction method over n, we obtain

$$\aleph_{n-a+1}^{(b)} = \sum_{k_b=1}^{n-a+1} \sum_{k_{b-1}=1}^{k_b} \cdots \sum_{k_1=1}^{k_2} \frac{1}{N_{k_1}} = \sum_{k=a}^{n} \binom{n-k+b-1}{b-1} \frac{1}{N_{k-a+1}}.$$

It is clearly seen that the proof is true for $n=1$. Let's assume that the proof is true for $n>1$. In that case;

$$\begin{aligned}
&\sum_{k_b=1}^{n-a+2} \sum_{k_{b-1}=1}^{k_b} \cdots \sum_{k_1=1}^{k_2} \frac{1}{N_{k_1}} = \sum_{k_b=1}^{n-a+1} \sum_{k_{b-1}=1}^{k_b} \cdots \sum_{k_1=1}^{k_2} \frac{1}{N_{k_1}} + \sum_{k_{b-1}=1}^{n-a+2} \sum_{k_{b-2}=1}^{k_{b-1}} \cdots \sum_{k_1=1}^{k_2} \frac{1}{N_{k_1}} \\
&= \sum_{k_b=1}^{n-a+1} \sum_{k_{b-1}=1}^{k_b} \cdots \sum_{k_1=1}^{k_2} \frac{1}{N_{k_1}} + \sum_{k_{b-1}=1}^{n-a+1} \sum_{k_{b-2}=1}^{k_{b-1}} \cdots \sum_{k_1=1}^{k_2} \frac{1}{N_{k_1}} + \sum_{k_{b-2}=1}^{n-a+2} \sum_{k_{b-3}=1}^{k_{b-2}} \cdots \sum_{k_1=1}^{k_2} \frac{1}{N_{k_1}} \\
&= \sum_{k_b=1}^{n-a+1} \sum_{k_{b-1}=1}^{k_b} \cdots \sum_{k_1=1}^{k_2} \frac{1}{N_{k_1}} + \sum_{k_{b-1}=1}^{n-a+1} \sum_{k_{b-2}=1}^{k_{b-1}} \cdots \sum_{k_1=1}^{k_2} \frac{1}{N_{k_1}} + \cdots + \sum_{k_2=1}^{n-a+1} \sum_{k_1=1}^{k_2} \frac{1}{N_{k_1}} + \sum_{k_1=1}^{n-a+2} \frac{1}{N_{k_1}} \\
&= \sum_{k_b=1}^{n-a+1} \sum_{k_{b-1}=1}^{k_b} \cdots \sum_{k_1=1}^{k_2} \frac{1}{N_{k_1}} + \sum_{k_{b-1}=1}^{n-a+1} \sum_{k_{b-2}=1}^{k_{b-1}} \cdots \sum_{k_1=1}^{k_2} \frac{1}{N_{k_1}} + \cdots + \sum_{k_2=1}^{n-a+1} \sum_{k_1=1}^{k_2} \frac{1}{N_{k_1}} + \sum_{k_1=1}^{n-a+1} \frac{1}{N_{k_1}} \\
&\quad + \frac{1}{N_{n-a+2}} \\
&= \sum_{k=a}^{k} \frac{1}{N_{n-a+1}} \left[\binom{n-k+b-1}{b-1} + \binom{n-k+b-2}{b-2} + \cdots + \binom{n-k}{0} \right] + \frac{1}{N_{n-a+2}} \\
&= \sum_{k=a}^{k} \frac{1}{N_{n-a+1}} \left[\binom{n-k+b}{b-1} \right] + \frac{1}{N_{n-a+2}} \\
&= \sum_{k=a}^{k+1} \frac{1}{N_{n-a+1}} \left[\binom{n-k+b}{b-1} \right]
\end{aligned}$$

as desired.

Corollary 2.13. We obtain the following result from Theorem 2.12

$$\aleph_n^{(r)} = \sum_{k=1}^{n} \binom{n-k+r-1}{r-1} \frac{1}{N_k}.$$

Proof. If we take $a=1$ *and* $b=r$ in the equation obtained from Theorem 2.12, we get

$$\aleph_n^{(r)} = \sum_{k=a}^{n} \binom{n-k+r-1}{r-1} \frac{1}{N_k}$$

$$\aleph_n^{(r)} = \binom{n+r-2}{r-1} \frac{1}{N_1} + \binom{n+r-3}{r-1} \frac{1}{N_2} + \cdots + \binom{r-1}{r-1} \frac{1}{N_n}.$$

This is desired equality.

Proposition 2.14. From the definition of Hyper-harmonic Narayana numbers and Corollary 2.13, the following equation is obtained:

$$\sum_{k=1}^{n} \frac{k}{N_k} = (n+1)\aleph_n - \aleph_n^{(2)}.$$

Proof. If we take $r = 2$ from Corollary 2.13,

$$\begin{aligned}\aleph_n^{(2)} &= \sum_{k=1}^{n} (n-k+1)\frac{1}{N_k} \\ &= n\sum_{k=1}^{n} \frac{1}{N_k} - \sum_{k=1}^{n} \frac{k}{N_k} + \sum_{k=1}^{n} \frac{1}{N_k} \\ &= (n+1)\aleph_n - \aleph_n^{(2)}.\end{aligned}$$

as desired.

Theorem 2.15. For $0 \le s \le r-1$, we have

$$\aleph_n^{(r)} = \sum_{u=1}^{n} \binom{n+r-s-u-1}{r-s-1} \aleph_u^{(s)}.$$

Proof. If the proof is done by induction over n, it is easily seen that it is true for $n = 1$. Let's assume that the proof is true for $n-1$, so let's show that the proof is true for n.

From Equation (15), we get

$$\begin{aligned}\aleph_n^{(r)} &= \aleph_{n-1}^{(r)} + \aleph_n^{(r-1)} \\ &= \aleph_{n-1}^{(r-1)} + \aleph_n^{(r-2)} + \aleph_{n-1}^{(r)} \\ &\vdots \\ &= \aleph_n^{(s)} + \aleph_{n-1}^{(s+1)} + \aleph_{n-1}^{(s+2)} + \cdots + \aleph_{n-1}^{(r-2)} + \aleph_n^{(r-1)} + \aleph_{n-1}^{(r)}.\end{aligned}$$

Using the induction hypothesis, we obtain

$$\aleph_{n-1}^{(r)} = \sum_{u=1}^{n-1} \binom{n+r-s-u-2}{r-s-1} \aleph_u^{(s)}, \aleph_n^{(r-1)} = \sum_{u=1}^{n} \binom{n+r-s-u-2}{r-s-2} \aleph_u^{(s)}.$$

Thus,

$$\aleph_n^{(r)} = \sum_{u=1}^{n-1} \binom{n+r-s-u-2}{r-s-1} \aleph_u^{(s)} + \sum_{u=1}^{n} \binom{n+r-s-u-2}{r-s-2} \aleph_u^{(s)}.$$

Expanding the above expression, we get

$$\aleph_n^{(r)} = \left(\sum_{u=1}^{n-1} \binom{n+r-s-u-2}{r-s-1} + \sum_{u=1}^{n} \binom{n+r-s-u-2}{r-s-2} \right) \aleph_u^{(s)} + \aleph_n^{(s)}.$$

By Pascal's identity,

$$\binom{n+r-s-u-2}{r-s-1}+\binom{n+r-s-u-2}{r-s-2}=\binom{n+r-s-u-1}{r-s-1}.$$

Therefore,

$$\aleph_n^{(r)}=\sum_{u=1}^{n-1}\binom{n+r-s-u-1}{r-s-1}\aleph_u^{(s)}+\aleph_n^{(s)}.$$

Since

$$\aleph_n^{(s)}=\binom{r-s-1}{r-s-1}\aleph_n^{(s)},$$

we obtain

$$\aleph_n^{(r)}=\sum_{u=1}^{n-1}\binom{n+r-s-u-1}{r-s-1}\aleph_u^{(s)}.$$

Thus, the proof is completed.

3. NORM PROPERTIES OF HARMONIC NARAYANA-RELATED MATRICES

The reciprocal Narayana numbers defined in the previous section, called Harmonic Narayana numbers, were as follows:

$$\aleph_n=\sum_{k=1}^{n}\frac{1}{N_k},n\geq 1\,.$$

Since we will use some combinatoric sums of this sequence of numbers in detail in this section, we can consider them as follows.

For $u\in\mathbb{Z}^+\cup\{0\}$

$$\sum_{i=0}^{n-1}\binom{i}{u}\aleph_i=\binom{n}{u+1}\aleph_n-\sum_{i=0}^{n-1}\binom{i+1}{u+1}\frac{1}{N_{i+1}}.$$

We have expressed the relationship between Harmonic Narayana numbers and Harmonic numbers as follows:

$$\sum_{i=0}^{n-1}\frac{\aleph_i}{i+1}=H_n\aleph_n-\sum_{i=0}^{n-1}\frac{H_{i+1}}{N_{i+1}}.$$

Another important formulas related to Harmonic Narayana numbers are defined in Equation (8) and (9).

Theorem 3.1. For $u\in\mathbb{Z}^+\cup\{0\}$, spectral norm of the matrix

$$Z=Circ\left(\binom{0}{u}\aleph_0,\binom{1}{u}\aleph_1,\dots,\binom{n-1}{u}\aleph_{n-1}\right)$$

is

$$\|Z\|_2=\binom{n}{u+1}\aleph_n-\sum_{i=0}^{n-1}\binom{i+1}{u+1}\frac{1}{N_{i+1}}.$$

Proof. Matrix Z is a normal matrix since the following formula is satisfied for the matrix. This means that.

$$\left(\overline{Z^T}\right).Z = Z.\left(\overline{Z^T}\right).$$

Then the spectral norm of this matrix is equal to its spectral radius. That is to say:

$$max\{|\lambda_i(Z)|\} = p(Z).$$

Here $p(.)$ defines the spectral radius.

Also, the matrix Z is irreducible. If the matrix Z has non-negative entries and its spectral radius is equal to the spectral radius of Z, then Z is equal to its Perron roots.

More precisely, let Z be a non-negative square matrix. In this case a non-negative eigenvalue $\lambda_i(Z)$, that is less than the absolute value of any other eigenvalue of Z is called a Perron root.

Let ξ define all vectors of degree n. That is; $\xi = (1,1,\dots,1)^T$. Then

$$Z\xi = \left(\sum_{i=0}^{n-1}\binom{i}{u}\aleph_i\right)\xi.$$

Here it can be easily seen that $\sum_{i=0}^{n-1}\binom{i}{u}\aleph_i$ is an eigenvalue of Z. This implies the existence of a Perron value of the matrix Z. From Equation (11);

$$\|Z\|_2 = \binom{n}{u+1}\aleph_n - \sum_{i=0}^{n-1}\binom{i+1}{u+1}\frac{1}{N_{i+1}}$$

as desired.

Corollary 3.2. From Theorem 3.1, we get

$$\sqrt{\sum_{i=0}^{n-1}\binom{i}{u}^2{\aleph_i}^2} \le \binom{n}{u+1}\aleph_n - \sum_{i=0}^{n-1}\binom{i+1}{u+1}\frac{1}{N_{i+1}} \le \sqrt{n\sum_{i=0}^{n-1}\binom{i}{u}^2{\aleph_i}^2}.$$

Proof. Proof follows explicitly from the definition of inequality (3) given for the Euclidean and spectral norm in connection with the definition of anti-difference operator (7).

Theorem 3.3. The spectral norm for matrix

$$Y = Circ\left(\frac{\aleph_0}{1}, \frac{\aleph_1}{2}, \dots, \frac{\aleph_{n-1}}{n}\right)$$

is expressed as follows:

$$\|Y\|_2 = H_n\aleph_n - \sum_{i=0}^{n-1}\frac{H_{i+1}}{N_{i+1}}.$$

Proof. The proof is similar to Theorem 3.1.
Namely,

$$\|Y\|_2 = \sum_{i=0}^{n-1} \frac{\aleph_i}{i+1}$$

and from Equation (13), we get

$$\|Y\|_2 = H_n \aleph_n - \sum_{i=0}^{n-1} \frac{H_{i+1}}{N_{i+1}}.$$

Corollary 3.4. From Theorem 2.3, we get

$$\sqrt{\sum_{i=0}^{n-1} \left(\frac{\aleph_i}{i+1}\right)^2} \leq H_n \aleph_n - \sum_{i=0}^{n-1} \frac{H_{i+1}}{N_{i+1}} \leq \sqrt{n \sum_{i=0}^{n-1} \left(\frac{\aleph_i}{i+1}\right)^2}.$$

Proof. The proof is obtained from the definition of anti-difference operator and equations (8,9).

Lemma 3.5. The following expression is provided

$$\sum_{i=1}^{n} i\, \aleph_i{}^2 = \aleph_{n+1}^2 \frac{(n+1)^{\underline{2}}}{2} - \frac{1}{2} \sum_{i=1}^{n} \frac{(i+1)^{\underline{2}}}{N_{i+1}} \left(2\aleph_i + \frac{1}{N_{i+1}}\right).$$

Proof. From Equation (7), Let $s(i) = (\aleph_i)^2$ and $\Delta t(i) = i^{\underline{1}}$, we get

$$\Delta s(i) = \frac{1}{N_{i+1}} \left(2\aleph_i + \frac{1}{N_{i+1}}\right) \text{ and } t(i) = \frac{i^{\underline{2}}}{2}.$$

From Equation (7) we obtain desired results.

Lemma 3.6. For the Harmonic Narayana numbers, we get

$$\sum_{i=1}^{n-1} (n-i)\, \aleph_{n+1}^2 = \aleph_{2n}^2 \frac{(n+1)^{\underline{2}}}{2} - n\aleph_{n+1}^2 - \frac{1}{2} \sum_{i=1}^{n-1} \frac{(i+1)(2n-i)}{N_{n+i+1}} \left(2\aleph_{n+i} + \frac{1}{N_{n+i+1}}\right).$$

Proof. From Equation (7), let $s(i) = (\aleph_{n+i})^2$ and $\Delta t(i) = n - i$, then we get

$$\Delta s(i) = \frac{1}{N_{n+i+1}} \left(2\aleph_{n+i} + \frac{1}{N_{n+i+1}}\right) \text{ and } t(i) = ni - \frac{i^{\underline{2}}}{2}.$$

If we substitute the above data in Equation (7), the following equation is obtained

$$\sum_{i=1}^{n-1} (n-i)\, \aleph_{n+1}^2 = \aleph_{n+i}^2 \left(ni - \frac{(i)^{\underline{2}}}{2}\right) |_1^n - \sum_{i=1}^{n} \left(n(i+1) - \frac{(i+1)^{\underline{2}}}{2}\right) \left(\frac{1}{N_{n+i+1}} \left(2\aleph_{n+i} + \frac{1}{N_{n+i+1}}\right)\right).$$

If the necessary algebraic operations are performed here, the desired result is obtained.

Lemma 3.7. For the harmonic Narayana numbers, we obtain following equation:

$$\sum_{i=1}^{n-1} 2^{i-1} \aleph_i^2 = \aleph_n^2\, 2^{n-1} - \sum_{i=0}^{n-1} \frac{2^i}{N_{i+1}} \left(2\aleph_i + \frac{1}{N_{i+1}}\right).$$

Proof. From Equation (7), let $s(i) = (\aleph_i)^2$ and $\Delta t(i) = 2^{i-1}$, then we get

$\Delta s(i) = \frac{1}{N_{i+1}}\left(2\aleph_i + \frac{1}{N_{i+1}}\right)$ and $t(i) = 2^{i-1}$.

If we substitute the data given above in Equation (7), the desired result is obtained

$$\begin{aligned}\sum_{i=1}^{n-1} 2^{i-1}\aleph_i^2 &= \aleph_i^2 2^{i-1}|_{i=1}^{n} - \sum_{i=1}^{n-1} 2^i \frac{1}{N_{i+1}}\left(2\aleph_i + \frac{1}{N_{i+1}}\right)\\ &= \aleph_n^2 2^{n-1} - \aleph_1^2 - \sum_{i=0}^{n-1} 2^i \frac{1}{N_{i+1}}\left(2\aleph_i + \frac{1}{N_{i+1}}\right) + \left(\frac{1}{N_1}\left(2\aleph_0 + \frac{1}{N_1}\right)\right)\\ &= \aleph_n^2 2^{n-1} - \sum_{i=0}^{n-1} \frac{2^i}{N_{i+1}}\left(2\aleph_i + \frac{1}{N_{i+1}}\right).\end{aligned}$$

Definition 3.8. Let $M = [\aleph_k]_{k=1}^{n}$, $N = [\aleph_{k+l-1}]_{k,l=1}^{n}$ and $T = \left[\sqrt{\binom{k-1}{n-l}}\,\aleph_k\right]_{k=1}^{n}$.

The characterization of harmonic Narayana-valued inputs for special matrices is written as follows:

$$M = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 2 & 2 & 2 & \cdots & 2 \\ 3 & 3 & 3 & \cdots & 3 \\ \frac{7}{2} & \frac{7}{2} & \frac{7}{2} & \cdots & \frac{7}{2} \\ \frac{23}{6} & \frac{23}{6} & \frac{23}{6} & \cdots & \frac{23}{6} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \aleph_n & \aleph_n & \aleph_n & \cdots & \aleph_n \end{bmatrix} \tag{16}$$

$$N = \begin{bmatrix} \aleph_1 & \aleph_2 & \aleph_3 & \cdots & \aleph_n \\ \aleph_2 & \aleph_3 & \aleph_4 & \cdots & \aleph_{n+1} \\ \aleph_3 & \aleph_4 & \aleph_5 & \cdots & \aleph_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \aleph_n & \aleph_{n+1} & \aleph_{n+2} & \cdots & \aleph_{2n-1} \end{bmatrix} \tag{17}$$

and

$$T = \begin{bmatrix} 0 & \ldots & 0 & 0 & \aleph_1 \\ 0 & \ldots & 0 & \aleph_2 & \aleph_2 \\ 0 & \ldots & \aleph_3 & \sqrt{2}\aleph_3 & \aleph_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \aleph_n & \ldots & \sqrt{\binom{n-1}{2}}\,\aleph_n & \sqrt{\binom{n-1}{1}}\,\aleph_n & \aleph_n \end{bmatrix}. \tag{18}$$

Equation (16) and (17) give characterization of harmonic Narayana with special matrices. Some theorems will be given with the help of the matrices and difference operators mentioned above.

Theorem 3.9. The eigenvalues of the $n \times n$ matrices M are

$$\mu_1 = n\aleph_n - \sum_{i=0}^{n} \frac{i+1}{N_{i+1}}$$

and $\mu_m = 0, m = 2,3,\ldots,n$.

Proof. The eigenvalues of the matrix M are roots of the $|\mu I - M| = 0$ such that

$$|\mu I - M| = \begin{vmatrix} \mu-1 & -1 & -1 & \cdots & -1 \\ -2 & \mu-2 & -2 & \cdots & -2 \\ -3 & -3 & \mu-3 & \cdots & -3 \\ -\frac{7}{2} & -\frac{7}{2} & -\frac{7}{2} & \cdots & -\frac{7}{2} \\ -\frac{23}{6} & -\frac{23}{6} & -\frac{23}{6} & \cdots & -\frac{23}{6} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\aleph_n & -\aleph_n & -\aleph_n & \cdots & \mu-\aleph_n \end{vmatrix} = 0.$$

The following result is obtained from the property of the determinant

$$|\mu I - M| = \begin{vmatrix} \mu-1 & -\mu & -\mu & \cdots & -\mu \\ -2 & \mu & 0 & \cdots & 0 \\ -3 & 0 & \mu & \cdots & 0 \\ -\frac{7}{2} & 0 & 0 & \cdots & 0 \\ -\frac{23}{6} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\aleph_n & 0 & 0 & \cdots & \mu \end{vmatrix},$$

$$|\mu I - M| = \mu^{n-1}\left(\mu - 1 - 2 - 3 - \frac{7}{2} - \cdots - \aleph_n\right) = 0.$$

The eigenvalues of the M matrices are

$$\mu_1 = 1 + 2 + 3 + \frac{7}{2} + \cdots + \aleph_n = \sum_{i=1}^{n} \aleph_i = n\aleph_n - \sum_{i=0}^{n-1} \frac{i+1}{N_{i+1}}$$

and $\mu_m = 0, m = 2,3, \ldots, n.$

Theorem 3.10. we have following equation

$$\|M\|_E = \sqrt{n^2\aleph_n^2 - n\sum_{i=0}^{n-1} \frac{i+1}{N_{i+1}}\left(2\aleph_i + \frac{1}{N_{i+1}}\right)}.$$

Proof. From Equation (1) and (10), we obtain

$$\|M\|_E = \sqrt{n\sum_{i=1}^{n} \aleph_i^2} = \sqrt{n^2\aleph_n^2 - n\sum_{i=0}^{n-1} \frac{i+1}{N_{i+1}}\left(2\aleph_i + \frac{1}{N_{i+1}}\right)}.$$

Theorem 3.11. We have following equation

$$\|M\|_2 = \sqrt{n\aleph_n^2 - \sum_{i=0}^{n-1} \frac{i+1}{N_{i+1}}\left(2\aleph_i + \frac{1}{N_{i+1}}\right)}$$

Proof. The spectral norm of the matrix M is

$$\|M\|_2 = \sqrt{\underbrace{max}_{1\le i\le n} \mu_i(M^H M)}.$$

The eigenvalues of $M^H M$, where M^H denotes the conjugate transpose of M, are denoted by μ_i. Namely,

$$M^H M = \begin{bmatrix} \sum_{i=1}^{n} \aleph_i^2 & \sum_{i=1}^{n} \aleph_i^2 & \cdots & \sum_{i=1}^{n} \aleph_i^2 \\ \sum_{i=1}^{n} \aleph_i^2 & \sum_{i=1}^{n} \aleph_i^2 & \cdots & \sum_{i=1}^{n} \aleph_i^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{n} \aleph_i^2 & \sum_{i=1}^{n} \aleph_i^2 & \cdots & \sum_{i=1}^{n} \aleph_i^2 \end{bmatrix}.$$

The eigenvalues $M^H M$ can be found by performing operations like Theorem 3.9. Namely, $\mu_1 = \sum_{i=1}^{n} \aleph_i^2$ and $\mu_m = 0, m = 2,3, \dots n$.

From Equation (10), we get

$$\|M\|_2 = \sqrt{n\aleph_n^2 - \sum_{i=0}^{n-1} \frac{i+1}{N_{i+1}} \left(2\aleph_i + \frac{1}{N_{i+1}}\right)}.$$

Theorem 3.12. The Euclidean norm of the matrix $N$is

$$\|N\|_E = \sqrt{\begin{aligned} &\aleph_{n+1}^2 \frac{(n+1)^{\underline{2}}}{2} - \frac{1}{2}\sum_{i=1}^{n} \frac{(i+1)^{\underline{2}}}{N_{i+1}} \left(2\aleph_i + \frac{1}{N_{i+1}}\right) + \frac{(n+1)^{\underline{2}}}{2} \aleph_{2n}^2 - n\aleph_{n+1}^2 \\ &\qquad - \frac{1}{2}\sum_{i=1}^{n-1} \frac{(i+1)(2n-i)}{N_{n+i+1}} \left(2\aleph_{i+n} + \frac{1}{N_{i+n+1}}\right) \end{aligned}}.$$

Proof. From Equation (1), we get

$$\|N\|_E = \sqrt{\sum_{i=1}^{n} i\aleph_i^2 + \sum_{i=1}^{n-1} (n-i)\aleph_{n+i}^2}.$$

We obtain the desired result by from Lemma 3.5 and 3.6.

Theorem 3.13.

$$\|T\|_E = \sqrt{\aleph_n^2 2^{n-1} - \sum_{i=0}^{n-1} \frac{2^i}{N_{i+1}} \left(2\aleph_i + \frac{1}{N_{i+1}}\right)}.$$

Proof. From Equation (1) and Lemma 3.7, we obtain

$$\|T\|_E = \sqrt{\sum_{i=1}^{n} 2^{i-1}\aleph_i^2} = \sqrt{\aleph_n^2 2^{n-1} - \sum_{i=0}^{n-1} \frac{2^i}{N_{i+1}} \left(2\aleph_i + \frac{1}{N_{i+1}}\right)}.$$

4. THE NORMS AND EIGENVALUES OF R-CIRCULANT MATRICES FEATURING HYPER-HARMONIC NARAYANA NUMBERS

Theorem 4.1. Let $C_r^{(i)} = Circ\left(\aleph_0^{(i)}, \aleph_1^{(i)}, \dots, \aleph_{n-1}^{(i)}\right)$ be an $n \times n$, r-circulant matrix. The Euclidean norm of $C_r^{(i)}$ is

$$\left\|C_r^{(i)}\right\|_E = \sqrt{\frac{n}{2}(n+1+(n-1)|r|^2)\left(\aleph_n^{(i)}\right)^2 - \frac{1}{2}\sum_{k=0}^{n-1}(k+1)\left(2n+k(|r|^2-1)\right)\left(\aleph_{k+1}^{(i-1)}+2\aleph_k^{(i)}\right)\aleph_{k+1}^{(i-1)}}.$$

Proof. From Equation (1), we get

$$\left\|C_r^{(i)}\right\|_E = \sqrt{\sum_{k=0}^{n-1}(n-k)\left(\aleph_k^{(i)}\right)^2 + \sum_{k=0}^{n-1}k|r|^2\left(\aleph_k^{(i)}\right)^2} = \sqrt{\sum_{k=0}^{n-1}(n+k(|r|^2-1))\left(\aleph_k^{(i)}\right)^2}.$$

From Equation (7), let $s(k) = \left(\aleph_k^{(i)}\right)^2$ and $\Delta t(k) = n + k(|r|^2 - 1)$,

$$\Delta s(k) = s(k+1) - s(k) = \left(\aleph_{k+1}^{(i)}\right)^2 - \left(\aleph_k^{(i)}\right)^2 = \left(\aleph_{k+1}^{(i)} - \aleph_k^{(i)}\right)\left(\aleph_{k+1}^{(i)} + \aleph_k^{(i)}\right).$$

From Lemma 2.11, we get

$$\Delta s(k) = \aleph_{k+1}^{(i-1)}(\aleph_{k+1}^{(i-1)} + 2\aleph_k^{(i)})$$

and

$$t(k) = nk + \frac{(k)^{\underline{2}}}{2}(|r|^2 - 1).$$

From Equation (7), we obtain

$$\left\|C_r^{(i)}\right\|_E = \sqrt{\frac{n}{2}(n+1+(n-1)|r|^2)\left(\aleph_n^{(i)}\right)^2 - \frac{1}{2}\sum_{k=0}^{n-1}(k+1)\left(2n+k(|r|^2-1)\right)\left(\aleph_{k+1}^{(i-1)}+2\aleph_k^{(i)}\right)\aleph_{k+1}^{(i-1)}}.$$

Theorem 4.2. Let $C^{(i)} = Circ\left(\aleph_0^{(i)}, \aleph_1^{(i)}, \dots, \aleph_{k-1}^{(i)}\right)$ be an $n \times n$, circulant matrix. The spectral norm of $C^{(i)}$ is

$$\left\|C^{(i)}\right\|_2 = \aleph_{k-1}^{(i+1)}.$$

Proof. The proof is similar to the previous theorem.

Corollary 4.3. For the Euclidean norm of the matrix $C^{(i)} = Circ\left(\aleph_0^{(i)}, \aleph_1^{(i)}, \ldots, \aleph_{n-1}^{(i)}\right)$, we have

$$\aleph_{n-1}^{(i+1)} \leq \left\| C_r^{(i)} \right\|_E \leq \sqrt{n}\aleph_{n-1}^{(i+1)}.$$

Proof. The proof is clear from the previous theorem and the relation between the Euclidean and spectral norm (Equation (3)).

Corollary 4.4. For hyper-harmonic Narayana numbers, we get

$$\frac{1}{\sqrt{n}}\aleph_{n-1}^{(i+1)} \leq \sqrt{\sum_{s=0}^{n-1}\left(\aleph_s^{(i)}\right)^2} \leq \aleph_{n-1}^{(i+1)}.$$

Proof. Since this result is derived from the relation between the spectral and Euclidean norms of the ordinary circulant matrix $C_1^{(i)} = Circ\left(\aleph_0^{(i)}, \aleph_1^{(i)}, \ldots, \aleph_{n-1}^{(i)}\right)$, it corresponds to the case $r = 1$. Hence, the statement is not intended for arbitrary r-circulant matrices. The desired inequality follows from Corollary 4.3 and the Euclidean norm definition.

Theorem 4.5. Let $C_r^{(i)} = Circ\left(\aleph_0^{(i)}, \aleph_1^{(i)}, \ldots, \aleph_{n-1}^{(i)}\right)$ be an $n \times n$, r-circulant matrix.

(i) If $|r| \geq 1$, then

$$\frac{1}{\sqrt{n}}\aleph_{n-1}^{(i+1)} \leq \left\| C_r^{(i)} \right\|_2 \leq \sqrt{(n-1)r^2+1}\aleph_{n-1}^{(i+1)},$$

(ii) If $|r| < 1$, then

$$\frac{|r|}{\sqrt{n}}\aleph_{n-1}^{(i+1)} \leq \left\| C_r^{(i)} \right\|_2 \leq \sqrt{n}\aleph_{n-1}^{(i+1)}.$$

Proof. We can express the r-circulant matrix of Hyper-harmonic-Narayana numbers as follows:

$$C_r^{(i)} = \begin{bmatrix} \aleph_0^{(i)} & \aleph_1^{(i)} & \aleph_2^{(i)} & \ldots & \aleph_{n-2}^{(i)} & \aleph_{n-1}^{(i)} \\ r\aleph_{n-1}^{(i)} & \aleph_0^{(i)} & \aleph_1^{(i)} & \ldots & \aleph_{n-3}^{(i)} & \aleph_{n-2}^{(i)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r\aleph_2^{(i)} & r\aleph_3^{(i)} & r\aleph_4^{(i)} & \ldots & \aleph_0^{(i)} & \aleph_1^{(i)} \\ r\aleph_1^{(i)} & r\aleph_2^{(i)} & r\aleph_3^{(i)} & \ldots & r\aleph_{n-1}^{(i)} & \aleph_0^{(i)} \end{bmatrix}.$$

We have

$$\left\| C_r^{(i)} \right\|_E = \sqrt{\sum_{k=0}^{n-1}(n-k)\left(\aleph_k^{(i)}\right)^2 + \sum_{k=0}^{n-1}k|r|^2\left(\aleph_k^{(i)}\right)^2}.$$

(i) From Corollary 4.4, we get

$$\frac{1}{\sqrt{n}}\aleph_{n-1}^{(i+1)} \leq \sqrt{\sum_{k=0}^{n-1}\left(\aleph_k^{(i)}\right)^2} \leq \aleph_{n-1}^{(i+1)}. \tag{19}$$

Since $|r| \geq 1$ and by Equation (19), we obtain

$$\left\|C_r^{(i)}\right\|_E \geq \sqrt{\sum_{k=0}^{n-1}(n-k)\left(\aleph_k^{(i)}\right)^2 + \sum_{k=0}^{n-1}k\left(\aleph_k^{(i)}\right)^2} \geq \sqrt{n\sum_{k=0}^{n-1}\left(\aleph_k^{(i)}\right)^2} \geq \aleph_{n-1}^{(i+1)}.$$

From Equation (3), we get

$$\frac{1}{\sqrt{n}}\aleph_{n-1}^{(i+1)} \leq \left\|C_r^{(i)}\right\|_2.$$

Now let us define the matrices X and Y

$$X = \begin{bmatrix} \aleph_0^{(i)} & 1 & 1 & \dots & 1 & 1 \\ r & \aleph_0^{(i)} & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & r & \dots & \aleph_0^{(i)} & 1 \\ r & r & r & \dots & r & \aleph_0^{(i)} \end{bmatrix},$$

and

$$Y = \begin{bmatrix} \aleph_0^{(i)} & \aleph_1^{(i)} & \aleph_2^{(i)} & \dots & \aleph_{n-2}^{(i)} & \aleph_{n-1}^{(i)} \\ \aleph_{n-1}^{(i)} & \aleph_0^{(i)} & \aleph_1^{(i)} & \dots & \aleph_{n-3}^{(i)} & \aleph_{n-2}^{(i)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \aleph_2^{(i)} & \aleph_3^{(i)} & \aleph_4^{(i)} & \dots & \aleph_0^{(i)} & \aleph_1^{(i)} \\ \aleph_1^{(i)} & \aleph_2^{(i)} & \aleph_3^{(i)} & \dots & \aleph_{n-1}^{(i)} & \aleph_0^{(i)} \end{bmatrix}.$$

After careful analysis, it is evident that the elements of the $C_r^{(i)}$ matrix are identical to those obtained by performing the Hadamard product of the corresponding elements of the X and Y matrices. So that

$$C_r^{(i)} = X \circ Y.$$

From Equation (4), $\|X \circ Y\|_2 \leq u_1(X)v_1(Y)$.

Consequently, it follows that $u_1(X) = \sqrt{(n-1)|r|^2+1}$, where $\sqrt{(n-1)|r|^2+1}$ denotes a matrix containing the sum of squares of the elements in the nth row of the X matrix. Similarly, we have $v_1(Y) = \sqrt{\sum_{k=0}^{n-1}\left(\aleph_k^{(i)}\right)^2}$, where $\sqrt{\sum_{k=0}^{n-1}\left(\aleph_k^{(i)}\right)^2}$ represents a matrix comprising the sum of squares of the elements in the nth column of the Y matrix. These specific elements within the X and Y matrices are prominently highlighted in bold.

From Equation (4), and Corollary 4.4, we obtain

$$\left\|C_r^{(i)}\right\|_2 \leq \sqrt{(n-1)|r|^2+1}\sqrt{\sum_{k=0}^{n-1}\left(\aleph_k^{(i)}\right)^2}.$$

Thus, we have

$$\frac{1}{\sqrt{n}}\aleph_{n-1}^{(i+1)} \leq \left\|C_r^{(i)}\right\|_2 \leq \sqrt{(n-1)|r|^2+1}\sqrt{\sum_{k=0}^{n-1}\left(\aleph_k^{(i)}\right)^2} \leq \sqrt{(n-1)|r|^2+1}.\aleph_{n-1}^{(i+1)}.$$

(ii) From $|r|<1$ and Corollary 4.4, we obtain

$$\left\|C_r^{(i)}\right\|_E = \sqrt{\sum_{k=0}^{n-1}(n-k)\left(\aleph_k^{(i)}\right)^2+\sum_{k=0}^{n-1}k|r|^2\left(\aleph_k^{(i)}\right)^2} \geq \sqrt{\sum_{k=0}^{n-1}(n-k)|r|^2\left(\aleph_k^{(i)}\right)^2+\sum_{k=0}^{n-1}k|r|^2\left(\aleph_k^{(i)}\right)^2}$$

$$= |r|\sqrt{\sum_{k=0}^{n-1}\left(\aleph_k^{(i)}\right)^2} \geq |r|\aleph_{n-1}^{(i+1)}.$$

From (3),

$$\frac{|r|}{\sqrt{n}}\aleph_{n-1}^{(i+1)} \leq \left\|C_r^{(i)}\right\|_2.$$

On the other hand, for the matrices X and Y we defined before, if the top row of matrix X is taken this time and matrix Y is continued in the same way, the following expressions are obtained

$$u_1(X) \overset{|r|<1}{=} \sqrt{(n)1^2+\left(\aleph_0^{(i)}\right)^2} = \sqrt{n},$$

$$v_1(Y) = \sqrt{\sum_{k=0}^{n-1}\left(\aleph_k^{(i)}\right)^2}.$$

From Corollary 4.4 and Equation (4), we have

$$\left\|C_r^{(i)}\right\|_2 \leq \sqrt{n}\aleph_{n-1}^{(i+1)},$$

$$\frac{|r|}{\sqrt{n}}\aleph_{n-1}^{(i+1)} \leq \left\|C_r^{(i)}\right\|_2 \leq \sqrt{n}\aleph_{n-1}^{(i+1)}.$$

The desired result is therefore achieved.

Example: To illustrate Theorem 4.5, let $i=1$ and $n=4$. Then the first hyper-harmonic Narayana numbers are

$$\aleph_0^{(1)}=0, \aleph_1^{(1)}=1, \aleph_2^{(1)}=2, \aleph_3^{(1)}=3.$$

Thus, for $r=2$, we obtain the following r-circulant matrix:

$$C_2^{(1)} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 6 & 0 & 1 & 2 \\ 4 & 6 & 0 & 1 \\ 2 & 4 & 6 & 0 \end{bmatrix}.$$

By direct numerical calculation, its spectral norm is

$$\left\| C_2^{(1)} \right\|_2 \cong 10.3132.$$

On the other hands since $|r| = 2 \geq 1$, Theorem 4.5 gives

$$\frac{1}{\sqrt{n}} \aleph_{n-1}^{(i+1)} \leq \left\| C_r^{(i)} \right\|_2 \leq \sqrt{(n-1)|r|^2 + 1}. \aleph_{n-1}^{(i+1)}.$$

For $n = 4, i = 1, and\ r = 2$, we have $\aleph_3^{(2)} = 6$.
Hence,

$$\sqrt{(4-1)|2|^2 + 1}. \aleph_3^{(2)} = \sqrt{13}. 6 \cong 21.63$$

$$3 \leq 10.31 \leq 21.63.$$

This numerical example confirms the validity of the inequality in Theorem 4.5 for $n = 4, i = 1, and\ r = 2$.

4. CONCLUSIONS

In this study, we introduced the concepts of harmonic and hyper-harmonic Narayana numbers and investigated several of their fundamental properties. Various summation formulas, combinatorial identities, and recursive relations involving these sequences were derived. In addition, several structured matrices associated with hyper-harmonic Narayana numbers were constructed and analysed.

Furthermore, upper and lower bounds for the spectral norms of circulant and r-circulant matrices involving hyper-harmonic Narayana numbers were obtained by using matrix norm inequalities and combinatorial techniques. The obtained numerical examples confirmed the consistency of the theoretical estimates.

We believe that the results presented in this paper contribute to the growing literature on harmonic-type special number sequences and their associated matrix structures. Possible extensions of this work may include investigations of determinants, eigenvalue distributions, and other algebraic properties of matrices constructed from generalized Narayana-type sequences.

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DECLARATION OF ARTIFICIAL INTELLIGENCE (AI) USE

When drafting the manuscript, neither generative AI agents nor large language models (LLMs) were used in any aspect of the study.

CONFLICTS OF INTEREST

No conflict of interest was declared by the author.

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