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A note on Cullen Numbers and Their Representations as Products of Jacobsthal Numbers

Cullen Sayılarının Jacobsthal Sayılarının Çarpımı Olarak İfadesi Üzerine

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ABSTRACT

In this paper, we investigate the Diophantine equation $C_n = J_a J_b$ where $C_n = n \cdot 2^n + 1$denotes the $n-th$ Cullen number and J_n denotes the Jacobsthal sequence. Combining the explicit formula with quantitative growth bounds for the Jacobsthal numbers, we show that all solutions satisfy $a = 3$. As a consequence, the problem reduces to the equations $C_n = 2^b + 1$ or $C_n = 2^b - 1$, depending on the parity of b.

We then solve these equations explicitly. In particular, we prove that an infinite family of solutions arises when n is a power of 2, while the remaining case admits only a trivial solution. As a result, we obtain a complete classification of Cullen numbers that can be expressed as products of two Jacobsthal numbers. The methods employed in this work rely on elementary properties of linear recurrence sequences and exponential growth comparisons. These techniques may be adapted to study similar Diophantine equations involving other special number sequences defined by linear recurrences.

Keywords: ***Cullen numbers, Jacobsthal numbers, Integer sequences, Diophantine equations, Linear recurrence sequences***

Highlights

- This paper completely classifies Cullen numbers that can be represented as products of two Jacobsthal numbers.
- It is proved that every solution of the Diophantine equation $C_n = J_a J_b$ necessarily satisfies $a = b = n$.
- The problem is reduced to two parity-dependent equations using explicit formulas and growth inequalities for Jacobsthal numbers.
- The methods developed in this work can be adapted to related exponential Diophantine equations involving binary linear recurrence sequences.

Cullen Sayılarının Jacobsthal Sayılarının Çarpımı Olarak İfadesi Üzerine

ÖZ

Bu çalışmada, Cullen sayıları ile Jacobsthal sayıları arasındaki aritmetik ilişki $C_n = J_a J_b$ Diophantine denklemi üzerinden incelenmiştir. Burada $C_n = n \cdot 2^n + 1$ n'inci Cullen sayısını, J_n ise Jacobsthal dizisini göstermektedir. Jacobsthal sayılarının açık formülü ve büyüme özellikleri kullanılarak, söz konusu denklemin her çözümünün zorunlu olarak $a = 3$ koşulunu sağladığı gösterilmiştir. Bu sonuç yardımıyla problem, b'nin tek ya da çift olmasına bağlı olarak $C_n = 2^b + 1$ veya $C_n = 2^b - 1$ denklemlerinin çözümüne indirgenmiştir.

Bu denklemler ayrıntılı olarak çözümlenmiş ve n'in bir 2 kuvveti olması durumunda sonsuz bir çözüm ailesinin elde edildiği, diğer durumda ise yalnızca önemsiz bir çözümün ortaya çıktığı ispatlanmıştır. Böylece, iki Jacobsthal sayısının çarpımı biçiminde yazılabilen Cullen sayılarının tam bir sınıflandırması verilmiştir. Bu çalışmada kullanılan yöntemler, lineer rekürans dizilerinin temel özelliklerine ve üstel büyüme karşılaştırmalarına dayanmakta olup, benzer Diophantine problemlerin incelenmesine de uygulanabilir niteliktedir.

Anahtar Kelimeler: Cullen sayıları, Jacobsthal sayıları, Tamsayı dizileri, Diophantine denklemler, Lineer rekürans dizileri

Öne Çıkanlar

- Bu çalışmada, iki Jacobsthal sayısının çarpımı biçiminde yazılabilen Cullen sayılarının tam sınıflandırması elde edilmiştir.
- $\boldsymbol{C_n = J_a J_b}$ Diophantine denkleminin her çözümünün zorunlu olarak $a = b = n$ koşulunu sağladığı ispatlanmıştır.
- Jacobsthal sayılarının açık formülü ve büyüme eşitsizlikleri kullanılarak problem pariteye bağlı iki denkleme indirgenmiştir.
- Burada kullanılan yöntemlerin, lineer rekürans dizileri içeren benzer üstel Diophantine denklemlere uygulanabileceği gösterilmiştir.

I. INTRODUCTION

Special integer sequences defined by linear recurrences or exponential expressions play a fundamental role in number theory, particularly in the investigation of divisibility properties, representation problems, and Diophantine equations. Among such sequences, the Cullen numbers, introduced by J. Cullen in 1905, are defined by

$$C_n = n2^n + 1 \tag{1}$$

and have attracted sustained interest due to their remarkable arithmetic behavior and their connections with primality, pseudo primality, and exponential Diophantine equations [3,5,7]. Over the years, several generalizations and variations of Cullen numbers have been proposed, revealing close links with classical recurrence sequences such as the Fibonacci and Lucas numbers [1,2,6].

Parallel to this line of research, the Jacobsthal sequence $(\mathrm{J_n})_{n\geq 0}$, defined by the recurrence relation

$$J_n = J_{n-1} + 2J_{n-2}, \quad J_0 = 0, J_1 = 1, \tag{2}$$

has been studied from both combinatorial and algebraic perspectives. Jacobsthal numbers arise naturally in problems related to binary representations, recursive structures, and matrix methods, and they share several structural properties with Fibonacci-type sequences [8,9,10]. In particular, their representation properties and associated number-theoretic aspects have been investigated in detail by Horadam and subsequent authors, emphasizing their relevance within the broader theory of linear recurrence sequences [8,9].

Recent studies indicate that Cullen numbers interact with linear recurrence sequences in subtle and nontrivial ways. For example, Cullen and Woodall numbers that coincide with terms of Fibonacci or related sequences have been characterized under specific arithmetic conditions [1,6]. In addition, divisibility properties, greatest common divisors, and pseudoprime aspects of Cullen numbers have been examined extensively [7,11]. Further investigations have considered Cullen numbers exhibiting special divisibility phenomena, including connections with Riesel and Sierpiński-type problems, using tools from analytic and algebraic number theory [4].

Motivated by these developments, this paper investigates arithmetic relations between Cullen numbers and Jacobsthal numbers, with a particular emphasis on multiplicative representations and divisibility constraints. By studying equations involving Cullen numbers expressed as products of Jacobsthal numbers, we aim to complement existing results on Cullen numbers within the framework of binary linear recurrence sequences. This perspective contributes to a more systematic understanding of how Cullen numbers interact with Jacobsthal-type sequences and highlights further arithmetic features of these classical yet relatively less explored recurrence sequences.

II. PRELIMINARIES

In this section, we present the basic definitions and properties of Cullen numbers and Jacobsthal numbers that will be used throughout the paper. These preliminaries provide the necessary background for the Diophantine equations investigated in the subsequent sections.

Definition 1. The sequence of Cullen numbers $(C_n)_{n\geq 1}$ is defined by

$$C_n = n \cdot 2^n + 1.$$

Cullen numbers were introduced by J. Cullen in 1905 and have since been studied extensively due to their exponential growth and remarkable arithmetic properties. From the definition, it is clear that for every integer

$n \geq 1$, the inequality

$$2^n < C_n < (n+1) \cdot 2^n$$

holds. In particular, every Cullen number is odd. Indeed, by definition $C_n = n \cdot 2^n + 1$. For $n \geq 1$, the term 2^n is even, and hence $n2^n$ is also even regardless of the parity of n. Adding 1 to an even integer yields an odd integer, so $C_n \equiv 1(mod2)$. Moreover, since $C_n \sim n2^n$, the sequence grows exponentially with respect to n. This rapid growth plays a crucial role in bounding the size of possible solutions to Diophantine equations involving Cullen numbers, as it allows one to derive effective upper bounds by comparison arguments.

Definition 2. The Jacobsthal sequence $(J_n)_{n\geq 0}$ is defined by the linear recurrence relation

$$J_n = J_{n-1} + 2J_{n-2},$$

with initial values $J_0 = 0$ and $J_1 = 1$.

This recurrence relation has characteristic equation

$$x^2 - x - 2 = 0,$$

whose roots are 2 and −1. Consequently, Jacobsthal numbers admit the explicit formula

$$J_n = \frac{2^n - (-1)^n}{3},$$

for all $n \geq 0$.

From this closed form, it follows that $J_n > 0$ for all $n \geq 1$. Moreover, the sequence satisfies the inequalities

$$\frac{2^n - 1}{3} \leq J_n \leq \frac{2^n + 1}{3},$$

which show that Jacobsthal numbers grow exponentially with base 2.

III. MAIN THEOREMS

Lemma 1. Let $n \geq 1$, and suppose that a Cullen number $C_n = n.2^n + 1$ can be written as a product of two Jacobsthal numbers, say

$$C_n = J_a J_b$$

with $a \leq b$. Then necessarily $a = 3$.

Proof. The Jacobsthal sequence admits the closed formula

$$J_k = \frac{2^k - (-1)^k}{3}.$$

In particular, for k ≥ 2 one has the equalities

$$2^{k-2} < J_k < 2^{k-1},$$

so that $J_k \sim {}^{2^k}/_3$ as $k \to \infty$.

Assume that

$$n.2^n + 1 = J_a J_b, \quad a \leq b.$$

First, observe that every Cullen number is odd, since $n.2^n$ is even for all $n \geq 1$, and hence $C_n \equiv 1(mod2)$. Consequently, both J_a and J_b must be odd. From the initial values

$$J_1 = 1, \quad J_2 = 1, \quad J_3 = 3, \quad J_4 = 5, \quad J_5 = 11, \ldots,$$

it follows that J_1 and J_2 are equal to 1. If $a \leq 2$, then

$$C_n = J_a J_b = J_b,$$

which would imply that a Cullen number coincides with a single Jacobsthal number. However, the asymptotic estimates

$$C_n \sim n2^n \text{ and } J_b \sim \frac{2^b}{3}$$

show that this is impossible for $n \geq 1$, since the former contains the additional multiplicative factor n, which cannot be absorbed by adjusting b. Hence $a \geq 3$.

We now show that $a \geq 4$ leads to contradiction. Using the equalities above, we obtain

$$J_a J_b > 2^{a-2}.2^{b-2} = 2^{a+b-4}.$$

Thus,

$$n.2^n + 1 > 2^{a+b-4}.$$

On the other hand, from the upper bound $J_k < 2^{k-1}$ we derive

$$n.2^n + 1 < 2^{a+b-2}.$$

Combining these estimates yields

$$2^{a+b-4} < n2^n + 1 < 2^{a+b-2}.$$

Since $n2^n$ dominates the constant term 1 for $n \geq 1$, the above inequalities imply that

$$a + b = n + O(\log n).$$

In particular, $a + b$ differs from n by at most a logarithmic term.

If $a \geq 4$, then $J_a \geq J_4 = 5$. Hence

$$C_n = J_a J_b \geq 5J_b.$$

Using the asymptotic behavior $J_b \sim {2^b}/{3}$, we obtain

$$C_n \sim \frac{5}{3}2^b.$$

However, the relation $a + b = n + O(\log n)$ implies that b is essentially of order n. Consequently, the right-hand side grows strictly faster than $n2^n$ when $a \geq 4$, contradicting the defining relation $C_n = n.2^n + 1$.

Therefore, the case $a \geq 4$ is impossible. Since we have already excluded $a \leq 2$, the only remaining possibility is

$$a = 3.$$

This completes the proof.

The Jacobsthal case is of particular interest for several structural reasons. First, the Jacobsthal sequence is defined by a binary linear recurrence with characteristic roots 2 and -1, and hence exhibits exponential growth governed by the dominant root 2. This makes it naturally compatible with Cullen-type sequences of the form $C_n = n.2^n + 1$, whose growth is also driven by powers of 2. Consequently, the interaction between Cullen numbers and Jacobsthal numbers provides a canonical testing ground for studying multiplicative relations between exponential sequences arising from linear recurrences.

Moreover, Jacobsthal numbers occupy an intermediate position between purely exponential sequences and more general linear recurrence sequences: they admit a simple closed formula while retaining nontrivial oscillatory behavior through the term $(-1)^n$. This structural simplicity allows for precise asymptotic estimates and sharp growth comparisons, which are essential tools in the study of exponential Diophantine equations.

From a Diophantine perspective, investigating factorizations of Cullen numbers into Jacobsthal numbers contributes to the broader program of understanding when terms of distinct linear recurrences can coincide or satisfy multiplicative relations. Such problems are central in the theory of exponential Diophantine equations and frequently require delicate use of growth arguments and linear forms in logarithms.

For these reasons, the Jacobsthal case represents a natural and mathematically significant instance within the general framework of Cullen-type Diophantine problems.

Theorem 1. Let n and b be positive integers. If a Cullen number C_n can be written as a product of two Jacobsthal numbers, that is,

$$C_n = J_a J_b,$$

then $a = 3$ and one of the following cases holds:

i. If b is odd, then

$$C_n = 2^b + 1.$$

ii. If b is even, then

$$C_n = 2^b - 1.$$

Proof. In view of Lemma 1, any solution of the equation

$$C_n = J_a J_b,$$

satisfies $a = 3$. Therefore, we may write

$$C_n = J_3 J_b.$$

Using the explicit formula for Jacobsthal numbers,

$$J_k = \frac{2^k - (-1)^k}{3},$$

we obtain $J_3 = 3$. Hence,

$$C_n = 3.J_b.$$

If b is odd, then $(-1)^b = -1$ and thus

$$J_b = \frac{2^b+1}{3}.$$

Substituting into the above equation gives

$$C_n = 2^b + 1.$$

If b is even, then $(-1)^b = 1$ and hence

$$J_b = \frac{2^b-1}{3}.$$

In this case, we obtain

$$C_n = 2^b - 1.$$

IV. MAIN RESULTS AND CONSEQUENCES

In this section, we derive explicit solutions of the Diophantine equation

$$C_n = J_a J_b,$$

by combining the structural result of Theorem 1 with the defining formula of Cullen numbers.

By Theorem 1, any solution of the equation

$$C_n = J_a J_b,$$

necessarily satisfies $a = 3$. Moreover, the problem reduces to one of the following two cases:

- $C_n = 2^b + 1$, where b is an odd integer,
- $C_n = 2^b - 1$, where b is an even integer.

Since Cullen numbers are defined by $C_n = n \cdot 2^n + 1$,, we analyze these two cases separately.

Case 1: $C_n = 2^b + 1$

Assume that $n \cdot 2^n + 1 = 2^b + 1$. Then $n \cdot 2^n = 2^b$. This equality implies that the left-hand side is a power of 2. Consequently, n itself must be a power of 2. Let $n = 2^t$ for some integer $t \geq 0$. Substituting this expression into the equation gives $2^t \cdot 2^{2^t} = 2^b$, and hence $b = 2^t + t$. Therefore, this case yields an infinite family of solutions parameterized by t. Taking into account the parity condition from Theorem 1, we note that this case corresponds to b being odd. The expression $b = 2^t + t$ is odd if and only if t is odd (except for $t = 0$, which gives $b = 1$). Hence, all admissible solutions in this case are given by: $n = 1$ and $b = 1, n = 2^t$ and $b = 2^t + t$ for odd integers $t \geq 1$.

Case 2: $C_n = 2^b - 1$

Assume that $n \cdot 2^n + 1 = 2^b - 1$. Then $n \cdot 2^n = 2^b - 2 = 2 \cdot (2^{b-1} - 1)$. Since $2^{b-1} - 1$ is odd, the right-hand side is divisible by 2 but not by 4. On the other hand, the left-hand side $n \cdot 2^n$ is divisible by 2^{n+1}. This divisibility condition forces: $n = 1$. Substituting $n = 1$ into the equation yields $2 = 2^b - 2$ which implies $2^b = 4$ and hence $b = 2$. This value of b is even, in agreement with the assumptions of this case.

Thus, Case 2 admits a unique solution.

Corollary 1: The Diophantine equation

$$C_n = J_a J_b$$

has solutions only with $a = 3$. Moreover, all solutions are given as follows:

1. $n = 1, a = 3, b = 1$, yielding
$$C_1 = 3 = J_3 J_1.$$
2. $n = 1, a = 3, b = 2$, yielding
$$C_1 = 3 = J_3 J_2.$$
3. $n = 2^t, a = 3, b = 2^t + t$ for odd integers $t \geq 1$ yielding
$$C_{2^t} = 2^{2^t+t} + 1 = J_3 J_{2^t+t}.$$

This completes the classification of Cullen numbers that can be expressed as products of two Jacobsthal numbers.

Remark: The infinite family of solutions described in Case 1 can be made explicit. For instance, taking $t = 11$ yields

$$n = 2^{11} = 2048,$$

and

$$b = 2^{11} + 11 = 2059,$$

In this case, the corresponding Cullen number satisfies

$$C_{2048} = 2048.\, 2^{2048} + 1 = 2^{2059} + 1,$$

and hence

$$C_{2048} = J_3 J_{2059}.$$

This example illustrates the effectiveness of the general construction and shows that solutions occur at very large indices.

V. CONCLUSION

In this paper, we obtained a complete classification of Cullen numbers that can be expressed as products of two Jacobsthal numbers. The argument relies on exponential growth comparison, parity considerations, and structural properties of binary linear recurrence sequences.

The method can be adapted to Diophantine equations of the form

$$n2^n + 1 = U_m U_k,$$

where U_n is a non-degenerate binary linear recurrence sequence. Extending the present approach to other classical sequences such as Fibonacci, Lucas, Pell, or Jacobsthal–Lucas numbers constitutes a natural continuation of this work.

A closely related problem concerns Woodall numbers

$$W_n = n2^n - 1.$$

Although structurally similar to Cullen numbers, their distinct divisibility behavior suggests that multiplicative representations in terms of recurrence sequences may exhibit fundamentally different phenomena.

More broadly, problems of this type belong to the theory of exponential Diophantine equations involving linear recurrences. Future research may investigate higher multiplicative decompositions, perfect power representations, or generalizations to Cullen-type sequences of the form $nr^n + 1$. Such studies could strengthen the connection between Cullen-type numbers and the broader framework of exponential Diophantine analysis.

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